28. Let \( F(x) = f(x) - 95 \) for \( x \geq 1 \). Writing \( k \) for \( m + 95 \), the given condition becomes
\[
F(k + F(n)) = F(k) + n, \quad k \geq 96, n \geq 1. \tag{1}
\]
Thus for \( x, z \geq 96 \) and an arbitrary \( y \) we have \( F(x + y) + z = F(x + y + F(z)) = F(x + F(y + z)) = F(x) + F(y) + z \), and consequently \( F(x + y) = F(x) + F(y) \) whenever \( x \geq 96 \). Moreover, since then \( F(x + y) + F(96) = F(x + y + 96) = F(x) + F(y + 96) = F(x) + F(y) + F(96) \) for any \( x, y \), we obtain
\[
F(x + y) = F(x) + F(y), \quad x, y \in \mathbb{N}. \tag{2}
\]
It follows by induction that \( F(n) = nc \) for all \( n \), where \( F(1) = c \). Equation (1) becomes \( ck + c^2n = ck + n \), and yields \( c = 1 \). Hence \( F(n) = n \) and \( f(n) = n + 95 \) for all \( n \).

Finally, \( \sum_{k=1}^{19} f(k) = 96 + 97 + \cdots + 114 = 1995 \).

**Second solution.** First we show that \( f(n) > 95 \) for all \( n \). If to the contrary \( f(n) \leq 95 \), we have \( f(m) = n + f(m + 95 - f(n)) \), so by induction \( f(m) = kn + f(m + k(95 - f(n))) \geq kn \) for all \( k \), which is impossible. Now for \( m > 95 \) we have \( f(m + f(n) - 95) = n + f(m) \), and again by induction \( f(m + k(f(n) - 95)) = kn + f(m) \) for all \( m, n, k \). It follows that with \( n \) fixed,
\[
(\forall m) \lim_{k \to \infty} \frac{f(m + k(f(n) - 95))}{m + k(f(n) - 95)} = \frac{n}{f(n) - 95};
\]
hence
\[
\lim_{s \to \infty} \frac{f(s)}{s} = \frac{n}{f(n) - 95}.
\]
Hence \( \frac{n}{f(n) - 95} \) does not depend on \( n \), i.e., \( f(n) \equiv cn + 95 \) for some constant \( c \). It is easily checked that only \( c = 1 \) is possible.
4.37 Solutions to the Shortlisted Problems of IMO 1996

1. We have \( a^5 + b^5 - a^2b^2(a + b) = (a^3 - b^3)(a^2 - b^2) \geq 0 \), i.e. \( a^5 + b^5 \geq a^2b^2(a + b) \). Hence

\[
\frac{ab}{a^5 + b^5 + ab} \leq \frac{ab}{a^2b^2(a + b) + ab} = \frac{abc^2}{a^2b^2c^2(a + b) + abc^2} = \frac{c}{a + b + c}.
\]

Now, the left side of the inequality to be proved does not exceed \( \frac{c}{a+b+c} + \frac{b}{a+b+c} = 1 \). Equality holds if and only if \( a = b = c \).

2. Clearly \( a_1 > 0 \), and if \( p \neq a_1 \), we must have \( a_n < 0 \), \( |a_n| > |a_1| \), and \( p = -a_n \). But then for sufficiently large odd \( k \), \( -a_n^k = |a_n|^k > (n-1)|a_1|^k \), so that \( a_1^k + \cdots + a_n^k \leq (n-1)|a_1|^k - |a_n|^k < 0 \), a contradiction. Hence \( p = a_1 \).

Now let \( x > a_1 \). From \( a_1 + \cdots + a_n \geq 0 \) we deduce \( \sum_{j=2}^{n} (x - a_j) \leq (n-1) \left( x + \frac{a_1}{n-1} \right) \), so by the AM–GM inequality,

\[
(x-a_2) \cdots (x-a_n) \leq \left( x + \frac{a_1}{n-1} \right)^{n-1} \leq x^{n-1} + x^{n-2}a_1 + \cdots + a_1^{n-1}.
\] (1)

The last inequality holds because \( \binom{n-1}{r} \leq (n-1)^r \) for all \( r \geq 0 \). Multiplying (1) by \( (x-a_1) \) yields the desired inequality.

3. Since \( a_1 > 2 \), it can be written as \( a_1 = b+b^{-1} \) for some \( b > 0 \). Furthermore, \( a_1^2 - 2 = b^2 + b^{-2} \) and hence \( a_2 = (b^2 + b^{-2})(b + b^{-1}) \). We prove that

\[
a_n = (b + b^{-1}) (b^2 + b^{-2}) (b^4 + b^{-4}) \cdots (b^{2^{n-1}} + b^{-2^{n-1}})
\]

by induction. Indeed, \( \frac{a_{n+1}}{a_n} = \left( \frac{a_n}{a_{n-1}} \right)^2 - 2 = \left( b^{2^{n-1}} + b^{-2^{n-1}} \right)^2 - 2 = b^{2^n} + b^{-2^n} \).

Now we have

\[
\sum_{i=1}^{n} \frac{1}{a_i} = 1 + \frac{b}{b^2 + 1} + \frac{b^3}{(b^2 + 1)(b^4 + 1)} + \cdots + \frac{b^{2^n}}{(b^2 + 1)(b^4 + 1) \cdots (b^{2^n} + 1)}.
\] (1)

Note that \( \frac{1}{b}(a+2-\sqrt{a^2-4}) = 1 + \frac{1}{b} \); hence we must prove that the right side in (1) is less than \( \frac{1}{b} \). This follows from the fact that

\[
\frac{b^{2^k}}{(b^2 + 1)(b^4 + 1) \cdots (b^{2^k} + 1)} = \frac{1}{(b^2 + 1)(b^4 + 1) \cdots (b^{2^k-1} + 1)} - \frac{1}{(b^2 + 1)(b^4 + 1) \cdots (b^{2^k} + 1)}; \]

hence the right side in (1) equals \( \frac{1}{b} \left( 1 - \frac{1}{(b^2+1)(b^4+1) \cdots (b^{2^n}+1)} \right) \), and this is clearly less than \( 1/b \).
4. Consider the function
\[ f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}. \]
Since \( f \) is strictly decreasing from \(+\infty\) to 0 on the interval \((0, +\infty)\), there exists exactly one \( R > 0 \) for which \( f(R) = 1 \). This \( R \) is also the only positive real root of the given polynomial.
Since \( \ln x \) is a concave function on \((0, +\infty)\), Jensen’s inequality gives us
\[
\sum_{j=1}^{n} \frac{a_j}{A} \left( \ln \frac{A}{R^j} \right) \leq \ln \left( \sum_{j=1}^{n} \frac{a_j}{A} \cdot \frac{A}{R^j} \right) = \ln f(R) = 0.
\]
Therefore \( \sum_{j=1}^{n} a_j (\ln A - j \ln R) \leq 0 \), which is equivalent to \( A \ln A \leq B \ln R \), i.e., \( A^A \leq R^B \).

5. Considering the polynomials \( \pm P(\pm x) \) we may assume w.l.o.g. that \( a, b \geq 0 \). We have four cases:
\begin{enumerate}
\item \( c \geq 0, d \geq 0 \). Then \( |a| + |b| + |c| + |d| = a + b + c + d = P(1) \leq 1 \).
\item \( c \geq 0, d < 0 \). Then \( |a| + |b| + |c| + |d| = a + b + c - d = P(1) - 2P(0) \leq 3 \).
\item \( c < 0, d \geq 0 \). Then
\[
|a| + |b| + |c| + |d| = a + b - c + d = \frac{4}{3} P(1) - \frac{1}{3} P(-1) - \frac{8}{3} P(1/2) + \frac{8}{3} P(-1/2) \leq 7.
\]
\item \( c < 0, d < 0 \). Then
\[
|a| + |b| + |c| + |d| = a + b - c - d = \frac{5}{3} P(1) - 4P(1/2) + \frac{4}{3} P(-1/2) \leq 7.
\]
\end{enumerate}

Remark. It can be shown that the maximum of 7 is attained only for \( P(x) = \pm (4x^3 - 3x) \).

6. Let \( f(x), g(x) \) be polynomials with integer coefficients such that
\[
f(x)(x+1)^n + g(x)(x^n + 1) = k_0.
\]
Write \( n = 2^r m \) for \( m \) odd and note that \( x^n + 1 = (x^{2^r} + 1)B(x) \), where \( B(x) = x^{2^r(m-1)} - x^{2^r(m-2)} - \cdots - x^{2^r} + 1 \). Moreover, \( B(-1) = 1 \); hence \( B(x) - 1 = (x + 1)c(x) \) and thus
\[
R(x)B(x) + 1 = (B(x) - 1)^n = (x + 1)^n c(x)^n \tag{1}
\]
for some polynomials \( c(x) \) and \( R(x) \).
The zeros of the polynomial \( x^{2^r} + 1 \) are \( \omega_j \), with \( \omega_1 = \cos \frac{\pi}{2^r} + i \sin \frac{\pi}{2^r} \), and \( \omega_j = \omega^{2j-1} \) for \( 1 \leq j \leq 2^r \). We have
From (\ast) we also get \( f(\omega_j)(\omega_j + 1)^n = k_0 \) for \( j = 1, 2, \ldots, 2^n \). Since 
\( A = f(\omega_1)f(\omega_2) \cdots f(\omega_{2^n}) \) is a symmetric polynomial in \( \omega_1, \ldots, \omega_{2^n} \) with integer coefficients, \( A \) is an integer. Consequently, taking the product over 
\( j = 1, 2, \ldots, 2^n \) and using (2) we deduce that \( 2^n A = k_0^2 \) is divisible by \( 2^n = 2^{2m} \). Hence \( 2^m \mid k_0 \).

Furthermore, since \( \omega_j + 1 = (\omega_1 + 1)p_j(\omega_1) \) for some polynomial \( p_j \) with integer coefficients, (2) gives \( (\omega_1 + 1)^2 p(\omega_1) = 2 \), where \( p(x) = p_2(x) \cdots p_{2^n}(x) \) has integer coefficients. But then the polynomial \( (x + 1)^2 p(x) - 2 \) has a zero \( x = \omega_1 \), so it is divisible by its minimal polynomial \( x^2 + 1 \). Therefore

\[
(x + 1)^2 p(x) = 2 + (x^2 + 1)q(x)
\]

for some polynomial \( q(x) \). Raising (3) to the \( m \)th power we get \( (x + 1)^n p(x)^n = 2^m + (x^2 + 1)Q(x) \) for some polynomial \( Q(x) \) with integer coefficients. Now using (1) we obtain

\[
(x + 1)^n c(x)^n (x^2 + 1)Q(x) = (x^2 + 1)Q(x) + (x^2 + 1)Q(x)B(x)R(x) = (x + 1)^n p(x)^n - 2^m + (x^n + 1)Q(X)R(x).
\]

Therefore \( (x+1)^n f(x) + (x^n+1)g(x) = 2^m \) for some polynomials \( f(x), g(x) \) with integer coefficients, and \( k_0 = 2^m \).

7. We are given that \( f(x+a+b) - f(x+a) = f(x+b) - f(x) \), where \( a = 1/6 \) and \( b = 1/7 \). Summing up these equations for \( x, x+b, \ldots, x+6b \) we obtain

\[
f(x+a+1) - f(x+a) = f(x+1) - f(x).
\]

Summing up the new equations for \( x, x+a, \ldots, x+5a \) we obtain that

\[
f(x+2) - f(x+1) = f(x+1) - f(x).
\]

It follows by induction that \( f(x+n) - f(x) = n[f(x+1) - f(x)] \). If \( f(x+1) \neq f(x) \), then \( f(x+n) - f(x) \) will exceed in absolute value an arbitrarily large number for a sufficiently large \( n \), contradicting the assumption that \( f \) is bounded. Hence \( f(x+1) = f(x) \) for all \( x \).

8. Putting \( m = n = 0 \) we obtain \( f(0) = 0 \) and consequently \( f(f(n)) = f(n) \) for all \( n \). Thus the given functional equation is equivalent to

\[
f(m + f(n)) = f(m) + f(n), \quad f(0) = 0.
\]

Clearly one solution is \((\forall x) f(x) = 0\). Suppose \( f \) is not the zero function. We observe that \( f \) has nonzero fixed points (for example, any \( f(n) \) is a fixed point). Let \( a \) be the smallest nonzero fixed point of \( f \). By induction, each \( ka \) (\( k \in \mathbb{N} \)) is a fixed point too. We claim that all fixed points of \( f \) are of this form. Indeed, suppose that \( b = ka + i \) is a fixed point, where \( i < a \). Then
\[ b = f(b) = f(ka + i) = f(i + f(ka)) = f(i) + f(ka) = f(i) + ka; \]

hence \( f(i) = i. \) Hence \( i = 0. \)

Since the set of values of \( f \) is a set of its fixed points, it follows that for \( i = 0, 1, \ldots, a - 1, \) \( f(i) = ani \) for some integers \( ni \geq 0 \) with \( n0 = 0. \)

Let \( n = ka + i \) be any positive integer, \( 0 \leq i < a. \) As before, the functional equation gives us

\[ f(n) = f(ka + i) = f(i) + ka = (ni + k)a. \]

Besides the zero function, this is the general solution of the given functional equation. To verify this, we plug in \( m = ka + i, \) \( n = la + j \) and obtain

\[ f(m + f(n)) = f(ka + i + f(la + j)) = f((k + l + nj)a + i) = (k + l + nj + ni)a = f(m) + f(n). \]

9. From the definition of \( a(n) \) we obtain

\[ a(n) - a([n/2]) = \begin{cases} 
1 & \text{if } n \equiv 0 \text{ or } n \equiv 3 \pmod{4}; \\
-1 & \text{if } n \equiv 1 \text{ or } n \equiv 2 \pmod{4}. 
\end{cases} \]

Let \( n = bkb_{k-1} \ldots b_1b_0 \) be the binary representation of \( n, \) where we assume \( b_k = 1. \) If we define \( p(n) \) and \( q(n) \) to be the number of indices \( i = 0, 1, \ldots, k - 1 \) with \( b_i = b_{i+1} \) and the number of \( i = 0, 1, \ldots, k - 1 \) with \( b_i \neq b_{i+1} \) respectively, we get

\[ a(n) = p(n) - q(n). \quad (1) \]

(a) The maximum value of \( a(n) \) for \( n \leq 1996 \) is 9 when \( p(n) = 9 \) and \( q(n) = 0, \) i.e., in the case \( n = \overline{1111111111}_2 = 1023. \)

The minimum value is \(-10\) and is attained when \( p(n) = 0 \) and \( q(n) = 10, \) i.e., only for \( n = \overline{1010101010}_2 = 1365. \)

(b) From (1) we have that \( a(n) = 0 \) is equivalent to \( p(n) = q(n) = k/2. \)

Hence \( k \) must be even, and the \( k/2 \) indices \( i \) for which \( b_i = b_{i+1} \) can be chosen in exactly \( \binom{k}{k/2} \) ways. Thus the number of positive integers \( n < 2^{11} = 2048 \) with \( a(n) = 0 \) is equal to

\[ \binom{0}{0} + \binom{2}{1} + \binom{4}{2} + \binom{6}{3} + \binom{8}{4} + \binom{10}{5} = 351. \]

But five of these numbers exceed 1996; these are \( 2002 = \overline{1111101010}_2, \)
\( 2004 = \overline{1111101010}_2, \) \( 2006 = \overline{1111101010}_2, \) \( 2010 = \overline{1111101101}_2, \)
\( 2026 = \overline{1111101101}_2. \) Therefore there are 346 numbers \( n \leq 1996 \) for which \( a(n) = 0. \)

10. We first show that \( H \) is the common orthocenter of the triangles \( ABC \) and \( AQR. \)
Let \( G, G', H' \) be respectively the centroid of \( \triangle ABC \), the centroid of \( \triangle PBC \), and the orthocenter of \( \triangle PBC \). Since the triangles \( ABC \) and \( PBC \) have a common circumcenter, from the properties of the Euler line we get \( \overrightarrow{HH'} = 3\overrightarrow{GG'} = -\overrightarrow{AP} \). But \( \triangle AQR \) is exactly the image of \( \triangle PBC \) under translation by \( \overrightarrow{AP} \); hence the orthocenter of \( AQR \) coincides with \( H \). (Remark: This can be shown by noting that \( AHBQ \) is cyclic.)

Now we have that \( RH \perp AQ \); hence \( \angle AXH = 90^\circ = \angle AEH \). It follows that \( AXEH \) is cyclic; hence

\[
\angle EXQ = 180^\circ - \angle AHE = 180^\circ - \angle BCA = 180^\circ - \angle BPA = \angle PAQ
\]

(as oriented angles). Hence \( EX \parallel AP \).

11. Let \( X, Y, Z \) respectively be the feet of the perpendiculars from \( P \) to \( BC \), \( CA \), \( AB \). Examining the cyclic quadrilaterals \( AZPY \), \( BXPZ \), \( CYPX \), one can easily see that \( \angle XZY = \angle APB - \angle C \) and \( XY = PC \sin \angle C \).

The first relation gives that \( XYZ \) is isosceles with \( XY = XZ \), so from the second relation \( PB \sin \angle B = PC \sin \angle C \). Hence \( AB/PB = AC/PC \). This implies that the bisectors \( BD \) and \( CD \) of \( \angle ABP \) and \( \angle ACP \) divide the segment \( AP \) in equal ratios; i.e., they concur with \( AP \).

Second solution. Take that \( X, Y, Z \) are the points of intersection of \( AP, BP, CP \) with the circumscribed circle of \( ABC \) instead. We similarly obtain \( XY = XZ \). If we write \( AP \cdot PX = BP \cdot PY = CP \cdot PZ = k \), from the similarity of \( \triangle APC \) and \( \triangle ZPX \) we get

\[
\frac{AC}{XZ} = \frac{AP}{PZ} = \frac{AP \cdot CP}{k},
\]

i.e., \( XZ = \frac{k \cdot AC \cdot BP}{AP \cdot BP \cdot CP} \). It follows again that \( AC/AB = PC/PB \).

Third solution. Apply an inversion with center at \( A \) and radius \( r \), and denote by \( \overline{Q} \) the image of any point \( Q \). Then the given condition becomes \( \angle BCP = \angle CBP \), i.e., \( BP = PC \). But

\[
PB = \frac{r^2}{AP \cdot AB} PB,
\]

so \( AC/AB = PC/PB \).

Remark. Moreover, it follows that the locus of \( P \) is an arc of the circle of Apollonius through \( C \).
12. It is easy to see that $P$ lies on the segment $AC$. Let $E$ be the foot of the altitude $BH$ and $Y, Z$ the midpoints of $AC, AB$ respectively. Draw the perpendicular $HR$ to $FP$ ($R \in FP$). Since $Y$ is the circumcenter of $\triangle FCA$, we have $\angle FYA = 180^\circ - 2\angle A$. Also, $OFPY$ is cyclic; hence $\angle OPF = \angle OYF = 2\angle A - 90^\circ$. Next, $\triangle OZF$ and $\triangle HRF$ are similar, so $OZ/OF = HR/HF$. This leads to

$HR \cdot OF = HF \cdot OZ = \frac{1}{2}HF \cdot \frac{1}{2}HE \cdot HB = HE \cdot OY \implies HR/HE = OY/OF$. Moreover, $\angle EHR = \angle FOY$; hence the triangles $EHR$ and $FOY$ are similar. Consequently $\angle HPC = \angle HRE = \angle OYF = 2\angle A - 90^\circ$, and finally, $\angle FHP = \angle HPC + \angle HCP = \angle A$.

Second solution. As before, $\angle HFY = 90^\circ - \angle A$, so it suffices to show that $HP \perp FY$. The points $O, F, P, Y$ lie on a circle, say $\Omega_1$ with center at the midpoint $Q$ of $OP$. Furthermore, the points $F, Y$ lie on the nine-point circle $\Omega$ of $\triangle ABC$ with center at the midpoint $N$ of $OH$. The segment $FY$ is the common chord of $\Omega_1$ and $\Omega$, from which we deduce that $NQ \perp FY$. However, $NQ \parallel HP$, and the result follows.

Third solution. Let $H'$ be the point symmetric to $H$ with respect to $AB$. Then $H'$ lies on the circumcircle of $\triangle ABC$. Let the line $FP$ meet the circumcircle at $U, V$ and meet $H'B$ at $P'$. Since $OF \perp UV$, $F$ is the midpoint of $UV$. By the butterfly theorem, $F$ is also the midpoint of $PP'$. Therefore $\triangle H'FP' \cong FHP$; hence $\angle FHP = \angle F'HB = \angle A$.

Remark. It is possible to solve the problem using trigonometry. For example, $\frac{FZ}{ZF} = \frac{PK}{KP} = \frac{\sin(A-B)}{\cos C}$, where $K$ is on $CF$ with $PK \perp CF$. Then $\frac{CF}{KP} = \frac{\sin(A-B)}{\cos C} + \tan A$, from which one obtains formulas for $KP$ and $KH$. Finally, we can calculate $\tan \angle FHP = \frac{KP}{KH} = \cdots = \tan A$.

Second remark. Here is what happens when $BC \leq CA$. If $\angle A > 45^\circ$, then $\angle FHP = \angle A$. If $\angle A = 45^\circ$, the point $P$ escapes to infinity. If $\angle A < 45^\circ$, the point $P$ appears on the extension of $AC$ over $C$, and $\angle FHP = 180^\circ - \angle A$.

13. By the law of cosines applied to $\triangle CA_1B_1$, we obtain

$A_1B_1^2 = A_1C^2 + B_1C^2 - A_1C \cdot B_1C \geq A_1C \cdot B_1C$.

Analogously, $B_1C_1^2 \geq B_1A \cdot C_1A$ and $C_1A_1^2 \geq C_1B \cdot A_1B$, so that multiplying these inequalities yields

$A_1B_1^2 \cdot B_1C_1^2 \cdot C_1A_1^2 \geq A_1B \cdot A_1C \cdot B_1A \cdot B_1C \cdot C_1A \cdot C_1B$. \hspace{1cm} (1)$

Now, the lines $AA_1, BB_1, CC_1$ concur, so by Ceva’s theorem, $A_1B \cdot B_1C \cdot C_1A = AB_1 \cdot BC_1 \cdot CA_1$, which together with (1) gives the desired inequality. Equality holds if and only if $CA_1 = CB_1$, etc.
14. Let $a, b, c, d, e,$ and $f$ denote the lengths of the sides $AB$, $BC$, $CD$, $DE$, $EF$, and $FA$ respectively.

Note that $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$. Draw the lines $PQ$ and $RS$ through $A$ and $D$ perpendicular to $BC$ and $EF$ respectively ($P, R \in BC$, $Q, S \in EF$). Then $BF \ge PQ = RS$. Therefore $2BF \ge PQ + RS$, or

$$2BF \ge (a \sin B + f \sin C) + (c \sin C + d \sin B),$$

and similarly, $2BD \ge (c \sin A + b \sin B) + (e \sin B + f \sin A)$, $2DF \ge (e \sin C + d \sin A) + (a \sin A + b \sin C)$.

Next, we have the following formulas for the considered circumradii:

$$R_A = \frac{BF}{2 \sin A}, \quad R_C = \frac{BD}{2 \sin C}, \quad R_E = \frac{DF}{2 \sin E}.$$ 

It follows from (1) that

$$R_A + R_C + R_E \ge \frac{1}{4} a \left( \frac{\sin B}{\sin A} + \frac{\sin A}{\sin B} \right) + \frac{1}{4} b \left( \frac{\sin C}{\sin B} + \frac{\sin B}{\sin C} \right) + \cdots$$

$$\ge \frac{1}{2} (a + b + \cdots) = \frac{P}{2},$$

with equality if and only if $\angle A = \angle B = \angle C = 120^\circ$ and $FB \perp BC$ etc., i.e., if and only if the hexagon is regular.

**Second solution.** Let us construct points $A''$, $C''$, $E''$ such that $ABA''F$, $CDC''B$, and $EFE''D$ are parallelograms. It follows that $A'', C'', B$ are collinear and also $C'', E'', B$ and $E'', A'', F$. Furthermore, let $A'$ be the intersection of the perpendiculars through $F$ and $B$ to $FA''$ and $BA''$, respectively, and let $C'$ and $E'$ be analogously defined. Since $A'FA''B$ is cyclic with the diameter being $A'A''$ and since $\triangle FA''B \cong \triangle BAF$, it follows that $2R_A = A'A'' = x$.

Similarly, $2R_C = C'C'' = y$ and $2R_E = E'E'' = z$. We also have $AB = FA'' = y_a$, $AF = A''B = z_a$, $CD = C''B = z_c$, $CB = C''D = x_c$, $EF = E''D = x_e$, and $ED = E''F = y_e$. The original inequality we must prove now becomes

$$x + y + z \ge y_a + z_a + z_c + x_c + x_e + y_e.$$ (1)
We now follow and generalize the standard proof of the Erdős–Mordell inequality (for the triangle $A'C'E'$), which is what (1) is equivalent to when $A'' = C'' = E''$. We set $C''E' = a$, $A'E' = c$ and $A'C'' = e$. Let $A_1$ be the point symmetric to $A''$ with respect to the bisector of $\angle E'A'C''$. Let $F_1$ and $B_1$ be the feet of the perpendiculars from $A_1$ to $A'C''$ and $A'E'$, respectively. In that case, $A_1 F_1 = A''F = y_a$ and $A_1 B_1 = A''B = z_a$. We have

$$ax = A'A \cdot E'C' \geq 2S_{A'E'A_1C'} = 2S_{A'E'A_1} + 2S_{A'C'A_1} = cz_a + ey_a .$$

Similarly, $cy \geq ex_c + az_c$ and $ez \geq ay_e + cx_e$. Thus

$$x + y + z \geq \frac{c}{a} z_a + \frac{a}{c} z_c + \frac{e}{c} x_c + \frac{c}{e} x_e + \frac{a}{e} y_e + \frac{e}{a} y_a = \left( \frac{c}{a} + \frac{a}{c} \right) \left( \frac{z_a + z_c}{2} \right) + \left( \frac{c}{e} + \frac{e}{c} \right) \left( \frac{x_e + x_c}{2} \right) + \cdots . \tag{2}$$

Let us set $a_1 = \frac{c - x_c}{c}, c_1 = \frac{y_p - y_a}{y_p}, e_1 = \frac{c - x_e}{c}$. We note that $\triangle A''C''E'' \sim \triangle A'C'E'$ and hence $a_1/a = c_1/c = e_1/e = k$. Thus $(\frac{e - a}{a} - \frac{k}{a}) e_1 + (\frac{c - a}{a} - \frac{k}{a}) c_1 = k (\frac{ac - y_p}{c} + \frac{y_p}{e} - \frac{ax_c}{c} + \frac{ae}{e} - \frac{ay_e}{e} - \frac{ax_e}{e}) = 0$. Equation (2) reduces to

$$x + y + z \geq \left( \frac{c}{a} + \frac{a}{c} \right) \left( \frac{z_a + z_c}{2} \right) + \left( \frac{c}{e} + \frac{e}{c} \right) \left( \frac{x_e + x_c}{2} \right) + \left( \frac{a}{e} + \frac{e}{a} \right) \left( \frac{y_a + y_e}{2} \right) .$$

Using $c/a + a/c, e/c + c/e, a/e + e/a \geq 2$ we finally get $x + y + z \geq y_a + z_a + z_c + x_c + x_e + y_e$. Equality holds if and only if $a = c = e$ and $A'' = C'' = E'' = \text{center of } \triangle A'C'E'$, i.e., if and only if $ABCDEF$ is regular.

**Remark.** From the second proof it is evident that the Erdős–Mordell inequality is a special case of the problem. if $P_a, P_b, P_c$ are the feet of the perpendiculars from a point $P$ inside $\triangle ABC$ to the sides $BC, CA, AB$, and $P_aPP_bP'_a, P_bPP_cP'_b, P_cPP_aP'_c$ parallelograms, we can apply the problem to the hexagon $P_aPP_bP'_aP'_bP_cP'_c$ to prove the Erdős–Mordell inequality for $\triangle ABC$ and point $P$.

15. Denote by $ABCD$ and $EFGH$ the two rectangles, where $AB = a, BC = b, EF = c$, and $FG = d$. Obviously, the first rectangle can be placed within the second one with the angle $\alpha$ between $AB$ and $EF$ if and only if

$$a \cos \alpha + b \sin \alpha \leq c, \quad a \sin \alpha + b \cos \alpha \leq d . \tag{1}$$

Hence $ABCD$ can be placed within $EFGH$ if and only if there is an $\alpha \in [0, \pi/2]$ for which (1) holds.
The lines \( l_1(\alpha x + \beta y = c) \) and \( l_2(\beta x + \alpha y = d) \) and the axes \( x \) and \( y \) bound a region \( R \). By (1), the desired placement of the rectangles is possible if and only if \( R \) contains some point \((\cos \alpha, \sin \alpha)\) of the unit circle centered at the origin \((0,0)\). This in turn holds if and only if the intersection point \( L \) of \( l_1 \) and \( l_2 \) lies outside the unit circle. It is easily computed that \( L \) has coordinates \( \left( \frac{bd - ac}{b^2 - a^2}, \frac{bc - ad}{b^2 - a^2} \right) \). Now \( L \) being outside the unit circle is exactly equivalent to the inequality we want to prove.

**Remark.** If equality holds, there is exactly one way of placing. This happens, for example, when \((a, b) = (5, 20)\) and \((c, d) = (16, 19)\).

**Second remark.** This problem is essentially very similar to (SL89-2).

16. Let \( A_1 \) be the point of intersection of \( OA' \) and \( BC \); similarly define \( B_1 \) and \( C_1 \). From the similarity of triangles \( OBA_1 \) and \( OA'B \) we obtain \( OA_1 \cdot OB_1 \cdot OC_1 = R^2 \). Now it is enough to show that \( 8OA_1 \cdot OB' \cdot OC' \leq R^3 \). Thus we must prove that

\[
\lambda \mu \nu \leq \frac{1}{8}, \quad \text{where} \quad \frac{OA_1}{OA} = \lambda, \quad \frac{OB_1}{OB} = \mu, \quad \frac{OC_1}{OC} = \nu. \quad (1)
\]

On the other hand, we have

\[
\frac{\lambda}{1 + \lambda} + \frac{\mu}{1 + \mu} + \frac{\nu}{1 + \nu} = \frac{S_{OBC}}{S_{ABC}} + \frac{S_{AOC}}{S_{ABC}} + \frac{S_{ABO}}{S_{ABC}} = 1.
\]

Simplifying this relation, we get

\[
1 = \lambda \mu + \mu \nu + \nu \lambda + 2 \lambda \mu \nu \geq 3(\lambda \mu \nu)^{2/3} + 2 \lambda \mu \nu,
\]

which cannot hold if \( \lambda \mu \nu > \frac{1}{8} \). Hence \( \lambda \mu \nu \leq \frac{1}{8} \), with equality if and only if \( \lambda = \mu = \nu = \frac{1}{2} \). This implies that \( O \) is the centroid of \( ABC \), and consequently, that the triangle is equilateral.

**Second solution.** In the official solution, the inequality to be proved is transformed into

\[
\cos(A - B) \cos(B - C) \cos(C - A) \geq 8 \cos A \cos B \cos C.
\]

Since \( \frac{\cos(B - C)}{\cos A} = -\frac{\cos(B - C)}{\cos(B + C)} = \frac{\tan B \tan C + 1}{\tan B \tan C - 1} \), the last inequality becomes

\[
(xy + 1)(yz + 1)(zx + 1) \geq 8(xy - 1)(yz - 1)(zx - 1), \quad \text{where we write} \ x, y, z
\]

for \( \tan A, \tan B, \tan C \). Using the relation \( x + y + z = xyz \), we can reduce this inequality to

\[
(2x + y + z)(x + 2y + z)(x + y + 2z) \geq 8(x + y)(y + z)(z + x).
\]

This follows from the AM–GM inequality: \( 2x + y + z = (x + y) + (x + z) \geq 2\sqrt{(x + y)(x + z)} \), etc.
17. Let the diagonals $AC$ and $BD$ meet in $X$. Either $\angle AXB$ or $\angle AXD$ is greater than or equal to $90^\circ$, so we assume w.l.o.g. that $\angle AXB \geq 90^\circ$. Let $\alpha, \beta, \alpha', \beta'$ denote $\angle CAB, \angle ABD, \angle BDC, \angle DCA$. These angles are all acute and satisfy $\alpha + \beta = \alpha' + \beta'$. Furthermore,

$$R_A = \frac{AD}{2 \sin \beta}, \quad R_B = \frac{BC}{2 \sin \alpha}, \quad R_C = \frac{BC}{2 \sin \alpha'}, \quad R_D = \frac{AD}{2 \sin \beta'}.$$

Let $\angle B + \angle D = 180^\circ$. Then $A, B, C, D$ are concyclic and trivially $R_A + R_C = R_B + R_D$. Let $\angle B + \angle D > 180^\circ$. Then $D$ lies within the circumcircle of $ABC$, which implies that $\beta > \beta'$. Similarly $\alpha < \alpha'$, so we obtain $R_A < R_D$ and $R_C < R_B$. Thus $R_A + R_C < R_B + R_D$. Let $\angle B + \angle D < 180^\circ$. As in the previous case, we deduce that $R_A > R_D$ and $R_C > R_B$, so $R_A + R_C > R_B + R_D$.

18. We first prove the result in the simplest case. Given a 2-gon $ABA$ and a point $O$, let $a, b, c, h$ denote $OA, OB, AB$, and the distance of $O$ from $AB$. Then $D = a + b$, $P = 2c$, and $H = 2h$, so we should show that

$$(a + b)^2 \geq 4h^2 + c^2. \quad (1)$$

Indeed, let $l$ be the line through $O$ parallel to $AB$, and $D$ the point symmetric to $B$ with respect to $l$. Then $(a + b)^2 = (OA + OB)^2 = (OA + OD)^2 \geq AD^2 = c^2 + 4h^2$.

Now we pass to the general case. Let $A_1A_2\ldots A_n$ be the polygon $\mathcal{F}$ and denote by $d_i, p_i$, and $h_i$ respectively $OA_i, A_iA_{i+1}$, and the distance of $O$ from $A_iA_{i+1}$ (where $A_{n+1} = A_1$). By the case proved above, we have for each $i, d_i + d_{i+1} \geq \sqrt{4h_i^2 + p_i^2}$. Summing these inequalities for $i = 1, \ldots, n$ and squaring, we obtain

$$4D^2 \geq \left(\sum_{i=1}^{n} \sqrt{4h_i^2 + p_i^2}\right)^2.$$ 

It remains only to prove that $\sum_{i=1}^{n} \sqrt{4h_i^2 + p_i^2} \geq \sqrt{\sum_{i=1}^{n} (4h_i^2 + p_i^2)} = \sqrt{4H^2 + D^2}$. But this follows immediately from the Minkowski inequality. Equality holds if and only if it holds in (1) and in the Minkowski inequality, i.e., if and only if $d_1 = \ldots = d_n$ and $h_1/p_1 = \ldots = h_n/p_n$. This means that $\mathcal{F}$ is inscribed in a circle with center at $O$ and $p_1 = \ldots = p_n$, so $\mathcal{F}$ is a regular polygon and $O$ its center.

19. It is easy to check that after 4 steps we will have all $a, b, c, d$ even. Thus $|ab - cd|, |ac - bd|, |ad - bc|$ remain divisible by 4, and clearly are not prime. The answer is no.

Second solution. After one step we have $a + b + c + d = 0$. Then $ac - bd = ac + b(a + b + c) = (a + b)(b + c)$ etc., so

$$|ab - cd| \cdot |ac - bd| \cdot |ad - bc| = (a + b)^2(a + c)^2(b + c)^2.$$
However, the product of three primes cannot be a square, hence the answer is no.

20. Let $15a + 16b = x^2$ and $16a - 15b = y^2$, where $x, y \in \mathbb{N}$. Then we obtain

$$x^4 + y^4 = (15a + 16b)^2 + (16a - 15b)^2 = (15^2 + 16^2)(a^2 + b^2) = 481(a^2 + b^2).$$

In particular, $481 = 13 \cdot 37 \mid x^4 + y^4$. We have the following lemma.

**Lemma.** Suppose that $p \mid x^4 + y^4$, where $x, y \in \mathbb{Z}$ and $p$ is an odd prime, where $p \not\equiv 1 \pmod{8}$. Then $p \mid x$ and $p \mid y$.

**Proof.** Since $p \mid x^8 - y^8$ and by Fermat’s theorem $p \mid x^{p-1} - y^{p-1}$, we deduce that $p \mid x^d - y^d$, where $d = (p - 1, 8)$. But $d \neq 8$, so $d \mid 4$. Thus $p \mid x^4 - y^4$, which implies that $p \mid 2y^4$, i.e., $p \mid y$ and $p \mid x$.

In particular, we can conclude that $13 \mid x, y$ and $37 \mid x, y$. Hence $x$ and $y$ are divisible by 481. Thus each of them is at least 481. On the other hand, $x = y = 481$ is possible. It is sufficient to take $a = 31 \cdot 481$ and $b = 481$.

**Second solution.** Note that $15x^2 + 16y^2 = 481a^2$. It can be directly verified that the divisibility of $15x^2 + 16y^2$ by 13 and by 37 implies that both $x$ and $y$ are divisible by both primes. Thus $481 \mid x, y$.

21. (a) It clearly suffices to show that for every integer $c$ there exists a quadratic sequence with $a_0 = 0$ and $a_n = c$, i.e., that $c$ can be expressed as $\pm 1^2 \pm 2^2 \pm \cdots \pm n^2$. Since

$$(n + 1)^2 - (n + 2)^2 - (n + 3)^2 + (n + 4)^2 = 4,$$

we observe that if our claim is true for $c$, then it is also true for $c \pm 4$. Thus it remains only to prove the claim for $c = 0, 1, 2, 3$. But one immediately finds $1 = 1^2$, $2 = -1^2 - 2^2 - 3^2 + 4^2$, and $3 = -1^2 + 2^2$, while the case $c = 0$ is trivial.

(b) We have $a_0 = 0$ and $a_n = 1996$. Since $a_n \leq 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$, we get $a_{17} \leq 1785$, so $n \geq 18$. On the other hand, $a_{18}$ is of the same parity as $1^2 + 2^2 + \cdots + 18^2 = 2109$, so it cannot be equal to 1996. Therefore we must have $n \geq 19$. To construct a required sequence with $n = 19$, we note that $1^2 + 2^2 + \cdots + 19^2 = 2470 = 1996 + 2 \cdot 237$; hence it is enough to write 237 as a sum of distinct squares. Since $237 = 14^2 + 5^2 + 4^2$, we finally obtain

$$1996 = 1^2 + 2^2 + 3^2 - 4^2 - 5^2 + 6^2 + \cdots + 13^2 - 14^2 + 15^2 + \cdots + 19^2.$$

22. Let $a, b \in \mathbb{N}$ satisfy the given equation. It is not possible that $a = b$ (since it leads to $a^2 + 2 = 2a$), so we assume w.l.o.g. that $a > b$. Next, for $a > b = 1$ the equation becomes $a^2 = 2a$, and one obtains a solution $(a, b) = (2, 1)$.

Let $b > 1$. If $\left\lfloor \frac{a^2}{b} \right\rfloor = \alpha$ and $\left\lfloor \frac{b^2}{a} \right\rfloor = \beta$, then we trivially have $ab \geq \alpha \beta$. Since also $\frac{a^2 + b^2}{ab} \geq 2$, we obtain $\alpha + \beta \geq \alpha \beta + 2$, or equivalently
Let $a = b^2 + c$ for some $c > 0$. Now the given equation becomes
\[ b^3 + 2bc + \left\lfloor \frac{c^2}{b} \right\rfloor = \left\lfloor \frac{b^3 + 2b^2c + b^2 + c^2}{b^3 + bc} \right\rfloor + b^3 + bc, \]
which reduces to
\[ (c - 1)b + \left\lfloor \frac{c^2}{b} \right\rfloor = \left\lfloor \frac{b^2(c + 1) + c^2}{b^3 + bc} \right\rfloor. \tag{1} \]

If $c = 1$, then (1) always holds, since both sides are 0. We obtain a family of solutions $(a, b) = (n, n^2 + 1)$ or $(a, b) = (n^2 + 1, n)$. Note that the solution $(1, 2)$ found earlier is obtained for $n = 1$.

If $c > 1$, then (1) implies that $\frac{b^2(c+1)+c^2}{b^3+bc} \geq (c-1)b$. This simplifies to
\[ c^2(b^2 - 1) + b^2(c(b^2 - 2) - (b^2 + 1)) \leq 0. \tag{2} \]

Since $c \geq 2$ and $b^2 - 2 \geq 0$, the only possibility is $b = 2$. But then (2) becomes $3c^2 + 8c - 20 \leq 0$, which does not hold for $c \geq 2$.

Hence the only solutions are $(n, n^2 + 1)$ and $(n^2 + 1, n)$, $n \in \mathbb{N}$.

23. We first observe that the given functional equation is equivalent to
\[ 4f \left( \frac{(3m+1)(3n+1)-1}{3} \right) + 1 = (4f(m) + 1)(4f(n) + 1). \]

This gives us the idea of introducing a function $g : 3\mathbb{N}_0 + 1 \rightarrow 4\mathbb{N}_0 + 1$ defined as $g(x) = 4f \left( \frac{x-1}{3} \right) + 1$. By the above equality, $g$ will be multiplicative, i.e.,
\[ g(xy) = g(x)g(y) \quad \text{for all } x, y \in 3\mathbb{N}_0 + 1. \]

Conversely, any multiplicative bijection $g$ from $3\mathbb{N}_0 + 1$ onto $4\mathbb{N}_0 + 1$ gives us a function $f$ with the required property: $f(x) = \frac{g(3x+1)-1}{4}$.

It remains to give an example of such a function $g$. Let $P_1, P_2, Q_1, Q_2$ be the sets of primes of the forms $3k+1, 3k+2, 4k+1, 4k+3$, respectively.

It is well known that these sets are infinite. Take any bijection $h$ from $P_1 \cup P_2$ onto $Q_1 \cup Q_2$ that maps $P_1$ bijectively onto $Q_1$ and $P_2$ bijectively onto $Q_2$. Now define $g$ as follows: $g(1) = 1$, and for $n = p_1p_2 \cdots p_m$ (prime powers need not be different) define $g(n) = h(p_1)h(p_2) \cdots h(p_m)$. Note that $g$ is well-defined. Indeed, among the $p_i$’s an even number are of the form $3k+2$, and consequently an even number of $h(p_i)$’s are of the form $4k+3$. Hence the product of the $h(p_i)$’s is of the form $4k+1$. Also, it is obvious that $g$ is multiplicative. Thus, the defined $g$ satisfies all the required properties.

24. We shall work on the array of lattice points defined by $\mathcal{A} = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x \leq 19, 0 \leq y \leq 11\}$. Our task is to move from $(0, 0)$ to $(19, 0)$ via the points of $\mathcal{A}$ so that each move has the form $(x, y) \rightarrow (x + a, y + b)$, where $a, b \in \mathbb{Z}$ and $a^2 + b^2 = r$. 

(a) If \( r \) is even, then \( a + b \) is even whenever \( a^2 + b^2 = r \) \((a, b \in \mathbb{Z})\). Thus
the parity of \( x + y \) does not change after each move, so we cannot
reach \((19,0)\) from \((0,0)\).

If \( 3 \mid r \), then both \( a \) and \( b \) are divisible by 3, so if a point \((x, y)\) can
be reached from \((0,0)\), we must have \( 3 \mid x \). Since \( 3 \mid 19 \), we cannot get
to \((19,0)\).

(b) We have \( r = 73 = 8^2 + 3^2 \), so each move is either \((x, y) \rightarrow (x \pm 8, y \pm 3)\)
or \((x, y) \rightarrow (x \pm 3, y \pm 8)\). One possible solution is shown in Fig. 1.

(c) We have \( 97 = 9^2 + 4^2 \). Let us partition \( A \) as \( B \cup C \), where \( B = \{(x, y) \in A \mid 4 \leq y \leq 7\} \). It is easily seen that moves of the type
\((x, y) \rightarrow (x \pm 9, y \pm 4)\) always take us from the set \( B \) to \( C \) and vice
versa, while the moves \((x, y) \rightarrow (x \pm 4, y \pm 9)\) always take us from \( C \) to
\( C \). Furthermore, each move of the type \((x, y) \rightarrow (x \pm 9, y \pm 4)\) changes
the parity of \( x \), so to get from \((0,0)\) to \((19,0)\) we must have an odd
number of such moves. On the other hand, with an odd number of
such moves, starting from \( C \) we can end up only in \( B \), although
the point \((19,0)\) is not in \( B \). Hence, the answer is \( no \).

**Remark.** Part (c) can also be solved by examining all cells that can be
reached from \((0,0)\). All these cells are marked in Fig. 2.

Fig. 1

Fig. 2

25. Let the vertices in the bottom row be assigned an arbitrary coloring,
and suppose that some two adjacent vertices receive the same color. The
number of such colorings equals \( 2^n - 2 \). It is easy to see that then the
colors of the remaining vertices get fixed uniquely in order to satisfy the
requirement. So in this case there are \( 2^n - 2 \) possible colorings.

Next, suppose that the vertices in the bottom row are colored alternately
red and blue. There are two such colorings. In this case, the same must
hold for every row, and thus we get \( 2^n \) possible colorings.

It follows that the total number of considered colorings is \((2^n - 2) + 2^n =
2^{n+1} - 2\).

26. Denote the required maximum size by \( M_k(m, n) \). If \( m < \frac{n(n+1)}{2} \), then
trivially \( M = k \), so from now on we assume that \( m \geq \frac{n(n+1)}{2} \).

First we give a lower bound for \( M \). Let \( r = r_k(m, n) \) be the largest integer
such that \( r + (r + 1) + \cdots + (r + n - 1) \leq m \). This is equivalent to

\[ nr \leq m - \frac{n(n-1)}{2} \leq n(r + 1) \]

so \( r = \left[ \frac{m}{n} - \frac{n-1}{2} \right] \). Clearly no \( n \) elements
from \( \{r+1, r+2, \ldots, k\} \) add up to \( m \), so
We claim that \( M \) is actually equal to \( k - r_k(m,n) \). To show this, we shall prove by induction on \( n \) that if no \( n \) elements of a set \( S \subseteq \{1, 2, \ldots, k\} \) add up to \( m \), then \( |S| \leq k - r_k(m,n) \).

For \( n = 2 \) the claim is true, because then for each \( i = 1, \ldots, r_k(m,2) = \left\lfloor \frac{m-1}{2} \right\rfloor \) at least one of \( i \) and \( m - i \) must be excluded from \( S \). Now let us assume that \( n > 2 \) and that the result holds for \( n - 1 \). Suppose that \( S \subseteq \{1, 2, \ldots, k\} \) does not contain \( n \) distinct elements with the sum \( m \), and let \( x \) be the smallest element of \( S \). We may assume that \( x \leq r_k(m,n) \), because otherwise the statement is clear. Consider the set \( S' = \{y - x \mid y \in S, y \neq x\} \). Then \( S' \) is a subset of \( \{1, 2, \ldots, k - x\} \) no \( n - 1 \) elements of which have the sum \( m - nx \). Also, it is easily checked that \( n - 1 \leq m - nx - 1 \leq k - x \), so we may apply the induction hypothesis, which yields that

\[
|S| \leq 1 + k - x - r_k(m - nx, n - 1) = k - \left\lfloor \frac{m - x - n(n - 1)}{n - 1} \right\rfloor.
\] (2)

On the other hand, \( \left( \frac{m - x}{n - 1} - \frac{n}{2} \right) - r_k(m, n) = \frac{m - nx - n(n - 1)}{n(n - 1)} \geq 0 \) because \( x \leq r_k(m, n) \); hence (2) implies \( |S| \leq k - r_k(m,n) \) as claimed.

27. Suppose that such sets of points \( A, B \) exist.

First, we observe that there exist five points \( A, B, C, D, E \) in \( A \) such that their convex hull does not contain any other point of \( A \). Indeed, take any point \( A \in A \). Since any two points of \( A \) are at distance at least 1, the number of points \( X \in A \) with \( XA \leq r \) is finite for every \( r > 0 \). Thus it is enough to choose four points \( B, C, D, E \) of \( A \) that are closest to \( A \). Now consider the convex hull \( C \) of \( A, B, C, D, E \).

Suppose that \( C \) is a pentagon, say \( ABCDE \). Then each of the disjoint triangles \( ABC, ACD, ADE \) contains a point of \( B \). Denote these points by \( P, Q, R \). Then \( \triangle PQR \) contains some point \( F \in A \), so \( F \) is inside \( ABCDE \), a contradiction.

Suppose that \( C \) is a quadrilateral, say \( ABCD \), with \( E \) lying within \( ABCD \). Then the triangles \( ABE, BCE, CDE, DAE \) contain some points \( P, Q, R, S \) of \( B \) that form two disjoint triangles. It follows that there are two points of \( A \) inside \( ABCD \), which is a contradiction.

Finally, suppose that \( C \) is a triangle with two points of \( A \) inside. Then \( C \) is the union of five disjoint triangles with vertices in \( A \), so there are at least five points of \( B \) inside \( C \). These five points make at least three disjoint triangles containing three points of \( A \). This is again a contradiction.

It follows that no such sets \( A, B \) exist.

28. Note that w.l.o.g., we can assume that \( p \) and \( q \) are coprime. Indeed, otherwise it suffices to consider the problem in which all \( x_i \)'s and \( p,q \) are divided by \( \gcd(p,q) \).
Let $k, l$ be the number of indices $i$ with $x_{i+1} - x_i = p$ and the number of those $i$ with $x_{i+1} - x_i = -q$ ($0 \leq i < n$). From $x_0 = x_n = 0$ we get $kp = lq$, so for some integer $t > 1$, $k = qt$, $l = pt$, and $n = (p+q)t$.

Consider the sequence $y_i = x_{i+p+q} - x_i$, $i = 0, \ldots, n-p-q$. We claim that at least one of the $y_i$’s equals zero. We begin by noting that each $y_i$ is of the form $up - vq$, where $u + v = p + q$; therefore $y_i = (u + v)p - v(p+q) = (p-v)(p+q)$ is always divisible by $p+q$. Moreover, $y_{i+1} - y_i = (x_{i+p+q+1} - x_{i+p+q}) - (x_{i+1} - x_i)$ is 0 or $\pm(p+q)$. We conclude that if no $y_i$ is 0 then all $y_i$’s are of the same sign. But this is in contradiction with the relation $y_0 + y_{p+q} + \cdots + y_{n-p-q} = x_n - x_0 = 0$. Consequently some $y_i$ is zero, as claimed.

**Second solution.** As before we assume $(p, q) = 1$. Let us define a sequence of points $A_i(y_i, z_i)$ ($i = 0, 1, \ldots, n$) in $\mathbb{N}_0^2$ inductively as follows. Set $A_0 = (0, 0)$ and define $(y_{i+1}, z_{i+1})$ as $(y_i, z_i + 1)$ if $x_{i+1} = x_i + p$ and $(y_i + 1, z_i)$ otherwise. The points $A_i$ form a trajectory $L$ in $\mathbb{N}_0^2$ continuously moving upwards and rightwards by steps of length 1. Clearly, $x_i = pqz_i - qy_i$ for all $i$. Since $x_n = 0$, it follows that $(z_n, y_n) = (kq, kp)$, $k \in \mathbb{N}$. Since $y_n + z_n = n > p + q$, it follows that $k > 1$. We observe that $x_i = x_j$ if and only if $A_iA_j \parallel A_0A_n$. We shall show that such $i, j$ with $i < j$ and $(i, j) \neq (0, n)$ must exist.

If $L$ meets $A_0A_n$ in an interior point, then our statement trivially holds. From now on we assume the opposite. Let $P_{ij}$ be the rectangle with sides parallel to the coordinate axes and with vertices at $(ip, jq)$ and $((i + 1)p, (j + 1)q)$. Let $L_{ij}$ be the part of the trajectory $L$ lying inside $P_{ij}$. We may assume w.l.o.g. that the endpoints of $L_{00}$ lie on the vertical sides of $P_{00}$. Then there obviously exists $d \in \{1, \ldots, k-1\}$ such that the endpoints of $L_{dd}$ lie on the horizontal sides of $P_{dd}$. Consider the translate $L'_{dd}$ of $L_{dd}$ for the vector $-d(p, q)$. The endpoints of $L'_{dd}$ lie on the vertical sides of $P_{dd}$. Hence $L_{00}$ and $L'_{dd}$ have some point $X \neq A_0$ in common. The translate $Y$ of point $X$ for the vector $d(p, q)$ belongs to $L$ and satisfies $XY \parallel A_0A_n$.

29. Let the squares be indexed serially by the integers: $\ldots, -1, 0, 1, 2, \ldots$. When a bean is moved from $i$ to $i + 1$ or from $i + 1$ to $i$ for the first time, we may assign the index $i$ to it. Thereafter, whenever some bean is moved in the opposite direction, we shall assume that it is exactly the one marked by $i$, and so on. Thus, each pair of neighboring squares has a bean stuck between it, and since the number of beans is finite, there are only finitely pairs of neighboring squares, and thus finitely many squares on which moves are made. Thus we may assume w.l.o.g. that all moves occur between 0 and $l \in \mathbb{N}$ and that all beans exist at all times within $[0, l]$.

Defining $b_i$ to be the number of beans in the $i$th cell ($i \in \mathbb{Z}$) and $b$ the total number of beans, we define the semi-invariant $S = \sum_{i \in \mathbb{Z}} i^2b_i$. Since all moves occur above 0, the semi-invariant $S$ increases by 2 with each
move, and since we always have $S < b \cdot l^2$, it follows that the number of moves must be finite.

We now prove the uniqueness of the final configuration and the number of moves for some initial configuration $\{b_i\}$. Let $x_i \geq 0$ be the number of moves made in the $i$th cell ($i \in \mathbb{Z}$) during the game. Since the game is finite, only finitely many of $x_i$’s are nonzero. Also, the number of beans in cell $i$, denoted as $e_i$, at the end is

$$
(\forall i \in \mathbb{Z}) \ e_i = b_i + x_{i-1} + x_{i+1} - 2x_i \in \{0, 1\}.
$$

Thus it is enough to show that given $b_i \geq 0$, the sequence $\{x_i\}_{i \in \mathbb{Z}}$ of nonnegative integers satisfying (1) is unique.

Suppose the assertion is false, i.e., that there exists at least one sequence $b_i \geq 0$ for which there exist distinct sequences $\{x_i\}$ and $\{x'_i\}$ satisfying (1).

We may choose such a $\{b_i\}$ for which $\min\{\sum_{i \in \mathbb{Z}} x_i, \sum_{i \in \mathbb{Z}} x'_i\}$ is minimal (since $\sum_{i \in \mathbb{Z}} x_i$ is always finite). We choose any index $j$ such that $b_j > 1$. Such an index $j$ exists, since otherwise the game is over. Then one must make at least one move in the $j$th cell, which implies that $x_j, x'_j \geq 1$.

However, then the sequences $\{x_i\}$ and $\{x'_i\}$ with $x_j$ and $x'_j$ decreased by 1 also satisfy (1) for a sequence $\{b_i\}$ where $b_{j-1}, b_j, b_{j+1}$ is replaced with $b_{j-1} + 1, b_j - 2, b_{j+1} + 1$. This contradicts the assumption of minimal $\min\{\sum_{i \in \mathbb{Z}} x_i, \sum_{i \in \mathbb{Z}} x'_i\}$ for the initial $\{b_i\}$.

30. For convenience, we shall write $f^2, fg, \ldots$ for the functions $f \circ f, f \circ g, \ldots$.

We need two lemmas.

**Lemma 1.** If $f(x) \in S$ and $g(x) \in T$, then $x \in S \cap T$.

**Proof.** The given condition means that $f^3(x) = g^2(x)$ and $gf^2(x) = fg^2(x)$. Since $x \in S \cup T = U$, we have two cases:

- $x \in S$. Then $f^2(x) = g^2(x)$, which also implies $f^3(x) = f^2(x)$. Therefore $g g(x) = g f^2(x) = f^3(x) = g^2(x)$, and since $g$ is a bijection, we obtain $f g(x) = g f(x)$, i.e., $x \in T$.

- $x \in T$. Then $f g(x) = f g(x)$, so $g^2 f(x) = g f g(x)$. It follows that $f^3(x) = g^2 f(x) = g f g(x) = g f^2(x)$, and since $f$ is a bijection, we obtain $x \in S$.

Hence $x \in S \cap T$ in both cases. Similarly, $f(x) \in T$ and $g(x) \in S$ again imply $x \in S \cap T$.

**Lemma 2.** $f(S \cap T) = g(S \cap T) = S \cap T$.

**Proof.** By symmetry, it is enough to prove $f(S \cap T) = S \cap T$, or in other words that $f^{-1}(S \cap T) = S \cap T$. Since $S \cap T$ is finite, this is equivalent to $f(S \cap T) \subseteq S \cap T$.

Let $f(x) \in S \cap T$. Then if $g(x) \in S$ (since $f(x) \in T$), Lemma 1 gives $x \in S \cap T$; similarly, if $g(x) \in T$, then by Lemma 1, $x \in S \cap T$.

Now we return to the problem. Assume that $f(x) \in S$. If $g(x) \not\in S$, then $g(x) \in T$, so from Lemma 1 we deduce that $x \in S \cap T$. Then Lemma 2 claims that $g(x) \in S \cap T$ too, a contradiction. Analogously, from $g(x) \in S$ we are led to $f(x) \in S$. This finishes the proof.
4.38 Solutions to the Shortlisted Problems of IMO 1997

1. Let $ABC$ be the given triangle, with $\angle B = 90^\circ$ and $AB = m$, $BC = n$. For an arbitrary polygon $\mathcal{P}$ we denote by $w(\mathcal{P})$ and $b(\mathcal{P})$ respectively the total areas of the white and black parts of $\mathcal{P}$.

(a) Let $D$ be the fourth vertex of the rectangle $ABCD$. When $m$ and $n$ are of the same parity, the coloring of the rectangle $ABCD$ is centrally symmetric with respect to the midpoint of $AC$. It follows that $w(ABC) = \frac{1}{2}w(ABCD)$ and $b(ABC) = \frac{1}{2}b(ABCD)$; thus $f(m, n) = \frac{1}{2}|w(ABCD) - b(ABCD)|$. Hence $f(m, n)$ equals $\frac{1}{2}$ if $m$ and $n$ are both odd, and $0$ otherwise.

(b) The result when $m, n$ are of the same parity follows from (a). Suppose that $m > n$, where $m$ and $n$ are of different parity. Choose a point $E$ on $AB$ such that $AE = 1$. Since by (a) $|w(EBC) - b(EBC)| = f(m - 1, n) \leq \frac{1}{2}$, we have $f(m, n) \leq \frac{1}{2} + |w(EAC) - b(EAC)| \leq \frac{1}{2} + S(EAC) = \frac{1}{2} + \frac{n - 1}{2} = \frac{n}{2}$. Therefore $f(m, n) \leq \frac{1}{2} \min(m, n)$.

(c) Let us calculate $f(m, n)$ for $m = 2k + 1$, $n = 2k$, $k \in \mathbb{N}$. With $E$ defined as in (b), we have $BE = BC = 2k$. If the square at $B$ is w.l.o.g. white, $CE$ passes only through black squares. The white part of $\triangle EAC$ consists of $2k$ similar triangles with areas $\frac{i}{4} \cdot \frac{i^2}{2k+1} = \frac{i^2}{4k(2k+1)}$, where $i = 1, 2, \ldots, 2k$. The total white area of $EAC$ is

$$\frac{1}{4k(2k+1)}(1^2 + 2^2 + \cdots + (2k)^2) = \frac{4k + 1}{12}.$$  

Therefore the black area is $(8k - 1)/12$, and $f(2k + 1, 2k) = (2k - 1)/6$, which is not bounded.

2. For any sequence $X = (x_1, x_2, \ldots, x_n)$ let us define

$$\overline{X} = (1, 2, \ldots, x_1, 1, 2, \ldots, x_2, \ldots, 1, 2, \ldots, x_n).$$

Also, for any two sequences $A, B$ we denote their concatenation by $AB$. It clearly holds that $\overline{AB} = \overline{A} \overline{B}$. The sequences $R_1, R_2, \ldots$ are given by $R_1 = (1)$ and $R_n = R_{n-1}(n)$ for $n > 1$.

We consider the family of sequences $Q_{ni}$ for $n, i \in \mathbb{N}$, $i \leq n$, defined as follows:

$$Q_{n1} = (1), \quad Q_{nn} = (n), \quad \text{and} \quad Q_{ni} = Q_{n-1,i-1}Q_{n-1,i} \quad \text{if } 1 < i < n.$$  

These sequences form a Pascal-like triangle, as shown in the picture below:

$$Q_{1i} : \quad 1$$  
$$Q_{2i} : \quad 1 \quad 2$$  
$$Q_{3i} : \quad 1 \quad 2 \quad 3$$  
$$Q_{4i} : \quad 1 \quad 112 \quad 123 \quad 4$$  
$$Q_{5i} : \quad 1 \quad 1112 \quad 112123 \quad 1234 \quad 5$$
We claim that $R_n$ is in fact exactly $Q_n1Q_n2 \ldots Q nn$. Before proving this, we observe that $Q_{ni} = Q_{n-1,i-1}Q_{n-1,i}$. This follows by induction, because $Q_{ni} = Q_{n-1,i-1}Q_{n-2,i} = Q_{n-2,i-1}Q_{n-1,i}$ for $n \geq 3$, $i \geq 2$ (the cases $i = 1$ and $n = 1, 2$ are trivial). Now $R_1 = Q_{11}$ and

$$R_n = R_{n-1}(n) = Q_{n-1,1} \ldots Q_{n-1,n-1}(n) = Q_n1 \ldots Q_{n,n-1}Q_{n,n}$$

for $n \geq 2$, which justifies our claim by induction.

Now we know enough about the sequence $R_n$ to return to the question of the problem. We use induction on $n$ once again. The result is obvious for $n = 1$ and $n = 2$. Given any $n \geq 3$, consider the $k$th elements of $R_n$ from the left, say $u$, and from the right, say $v$. Assume that $u$ is a member of $Q_{nj}$, and consequently that $v$ is a member of $Q_{n,n+1-j}$. Then $u$ and $v$ come from symmetric positions of $R_{n-1}$ (either from $Q_{n-1,j}, Q_{n-1,n-j}$, or from $Q_{n-1,j-1}, Q_{n-1,n+1-j}$), and by the inductive hypothesis exactly one of them is 1.

3. (a) For $n = 4$, consider a convex quadrilateral $ABCD$ in which $AB = BC = AC = BD$ and $AD = DC$, and take the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$. For $n = 5$, take the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EA}$ for any regular pentagon $ABCDE$.

(b) Let us draw the vectors of $V$ as originated from the same point $O$. Consider any maximal subset $B \subset V$, and denote by $u$ the sum of all vectors from $B$. If $l$ is the line through $O$ perpendicular to $u$, then $B$ contains exactly those vectors from $V$ that lie on the same side of $l$ as $u$ does, and no others. Indeed, if any $v \notin B$ lies on the same side of $l$, then $|u + v| \geq |u|$; similarly, if some $v \in B$ lies on the other side of $l$, then $|u - v| \geq |u|$. Therefore every maximal subset is determined by some line $l$ as the set of vectors lying on the same side of $l$. It is obvious that in this way we get at most $2n$ sets.

4. (a) Suppose that an $n \times n$ coveralls matrix $A$ exists for some $n > 1$. Let $x \in \{1, 2, \ldots, 2n - 1\}$ be a fixed number that does not appear on the fixed diagonal of $A$. Such an element must exist, since the diagonal can contain at most $n$ different numbers. Let us call the union of the $i$th row and the $i$th column the $i$th cross. There are $n$ crosses, and each of them contains exactly one $x$. On the other hand, each entry $x$ of $A$ is contained in exactly two crosses. Hence $n$ must be even. However, 1997 is an odd number; hence no coveralls matrix exists for $n = 1997$.

(b) For $n = 2$, $A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ is a coveralls matrix. For $n = 4$, one such matrix is, for example,

$$A_4 = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 1 & 7 & 5 \\ 4 & 6 & 1 & 2 \\ 7 & 4 & 3 & 1 \end{bmatrix}.$$
This construction can be generalized. Suppose that we are given an \( n \times n \) coveralls matrix \( A_n \). Let \( B_n \) be the matrix obtained from \( A_n \) by adding \( 2n \) to each entry, and \( C_n \) the matrix obtained from \( B_n \) by replacing each diagonal entry (equal to \( 2n + 1 \) by induction) with \( 2n \). Then the matrix

\[
A_{2n} = \begin{bmatrix} A_n & B_n \\ C_n & A_n \end{bmatrix}
\]

is coveralls. To show this, suppose that \( i \leq n \) (the case \( i > n \) is similar). The \( i \)th cross is composed of the \( i \)th cross of \( A_n \), the \( i \)th row of \( B_n \), and the \( i \)th column of \( C_n \). The \( i \)th cross of \( A_i \) covers \( 1, 2, \ldots, 2n - 1 \). The \( i \)th row of \( B_n \) covers all numbers of the form \( 2n + j \), where \( j \) is covered by the \( i \)th row of \( A_n \) (including \( j = 1 \)). Similarly, the \( i \)th column of \( C_n \) covers \( 2n \) and all numbers of the form \( 2n + k \), where \( k > 1 \) is covered by the \( i \)th column of \( A_n \). Thus we see that all numbers are accounted for in the \( i \)th cross of \( A_{2n} \), and hence \( A_{2n} \) is a desired coveralls matrix. It follows that we can find a coveralls matrix whenever \( n \) is a power of 2.

**Second solution for part b.** We construct a coveralls matrix explicitly for \( n = 2^k \). We consider the coordinates/cells of the matrix elements modulo \( n \) throughout the solution. We define the \( i \)-diagonal (\( 0 \leq i < n \)) to be the set of cells of the form \( (j, j + i) \), for all \( j \). We note that each cross contains exactly one cell from the 0-diagonal (the main diagonal) and two cells from each \( i \)-diagonal. For two cells within an \( i \) diagonal, \( x \) and \( y \), we define \( x \) and \( y \) to be related if there exists a cross containing both \( x \) and \( y \). Evidently, for every cell \( x \) not on the 0-diagonal there are exactly two other cells related to it. The relation thus breaks up each \( i \)-diagonal (\( i > 0 \)) into cycles of length larger than 1. Due to the diagonal translational symmetry (modulo \( n \)), all the cycles within a given \( i \)-diagonal must be of equal length and thus of an even length, since \( n = 2^k \).

The construction of a coveralls matrix is now obvious. We select a number, say 1, to place on all the cells of the 0-diagonal. We pair up the remaining numbers and assign each pair to an \( i \)-diagonal, say \((2i, 2i+1)\). Going along each cycle within the \( i \)-diagonal we alternately assign values of \( 2i \) and \( 2i + 1 \). Since the cycle has an even length, a cell will be related only to a cell of a different number, and hence each cross will contain both \( 2i \) and \( 2i + 1 \).

5. We shall prove first the 2-dimensional analogue:

**Lemma.** Given an equilateral triangle \( ABC \) and two points \( M, N \) on the sides \( AB \) and \( AC \) respectively, there exists a triangle with sides \( CM, BN, MN \).

**Proof.** Consider a regular tetrahedron \( ABCD \). Since \( CM = DM \) and \( BN = DN \), one such triangle is \( DMN \).
Now, to solve the problem for a regular tetrahedron $ABCD$, we consider a 4-dimensional polytope $ABCDE$ whose faces $ABCD$, $ABCE$, $ABDE$, $ACDE$, $BCDE$ are regular tetrahedra. We don’t know what it looks like, but it yields a desired triangle: for $M \in ABC$ and $N \in ADC$, we have $DM = EM$ and $BN = EN$; hence the desired triangle is $EMN$.

**Remark.** A solution that avoids embedding in $\mathbb{R}^4$ is possible, but no longer so short.

6. (a) One solution is

$$x = 2^n 3^{n+1}, \quad y = 2^n - n3^n, \quad z = 2^n - 2n + 3^{n-1}.$$

(b) Suppose w.l.o.g. that $\gcd(c, a) = 1$. We look for a solution of the form

$$x = p^m, \quad y = p^n, \quad z = q^r, \quad p, q, m, n, r \in \mathbb{N}.$$

Then $x^a + y^b = p^{ma} + p^{nb}$ and $z^c = q^c p^r$, and we see that it is enough to assume $ma - 1 = nb = rc$ (there are infinitely many such triples $(m, n, r)$) and $q^c = p + 1$.

7. Let us set $AC = a$, $CE = b$, $EA = c$. Applying Ptolemy’s inequality for the quadrilateral $ACEF$ we get

$$AC \cdot EF + CE \cdot AF \geq AE \cdot CF.$$

Since $EF = AF$, this implies $\frac{FA}{FC} \geq \frac{c}{a + b}$. Similarly $\frac{BC}{BE} \geq \frac{a}{b + c}$ and $\frac{DE}{DA} \geq \frac{b}{c + a}$. Now,

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}.$$

Hence it is enough to prove that

$$\frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b} \geq \frac{3}{2}. \quad (1)$$

If we now substitute $x = b + c$, $y = c + a$, $z = a + b$ and $S = a + b + c$ the inequality (1) becomes equivalent to $S(1/x + 1/y + 1/z) - 3 \geq 3/2$ which follows immediately from $1/x + 1/y + 1/z \geq 9/(x + y + z) = 9/(2S)$. Equality occurs if it holds in Ptolemy’s inequalities and also $a = b = c$. The former happens if and only if the hexagon is cyclic. Hence the only case of equality is when $ABCDEF$ is regular.

8. (a) Denote by $b$ and $c$ the perpendicular bisectors of $AB$ and $AC$ respectively. If w.l.o.g. $b$ and $AD$ do not intersect (are parallel), then $\angle BCD = \angle BAD = 90^\circ$, a contradiction. Hence $V, W$ are well-defined. Now, $\angle DWB = 2\angle DAB$ and $\angle DVC = 2\angle DAC$ as oriented angles, and therefore $\angle(WB, VC) = 2(\angle DVC - \angle DWB) = 2\angle BAC = 2\angle BCD$ is not equal to 0. Consequently $CV$ and $BW$ meet at some $T$ with $\angle BTC = 2\angle BAC$. 


(b) Let $B'$ be the second point of intersection of $BW$ with $\Gamma$. Clearly $AD = BB'$. But we also have $\angle BTC = 2\angle BAC = 2\angle BB'C$, which implies that $CT = TB'$. It follows that $AD = BB' = |BT \pm TB'| = |BT \pm CT|$.

Remark. This problem is also solved easily using trigonometry.

9. For $i = 1, 2, 3$ (all indices in this problem will be modulo 3) we denote by $O_i$ the center of $C_i$ and by $M_i$ the midpoint of the arc $A_{i+1}A_{i+2}$ that does not contain $A_i$. First we have that $O_{i+1}O_{i+2}$ is the perpendicular bisector of $IB_i$, and thus it contains the circumcenter $R_i$ of $A_iB_iI$. Additionally, it is easy to show that $T_{i+1}A_i = T_{i+1}I$ and $T_{i+2}A_i = T_{i+2}I$, which implies that $R_i$ lies on the line $T_{i+1}T_{i+2}$. Therefore $R_i = O_{i+1}O_{i+2} \cap T_{i+1}T_{i+2}$.

Now, the lines $T_1O_1, T_2O_2, T_3O_3$ are concurrent at $I$. By Desargues’s theorem, the points of intersection of $O_{i+1}O_{i+2}$ and $T_{i+1}T_{i+2}$, i.e., the $R_i$’s, lie on a line for $i = 1, 2, 3$.

Second solution. The centers of three circles passing through the same point $I$ and not touching each other are collinear if and only if they have another common point. Hence it is enough to show that the circles $A_iB_iI$ have a common point other than $I$.

Now apply inversion at center $I$ and with an arbitrary power. We shall denote by $X'$ the image of $X$ under this inversion. In our case, the image of the circle $C_i$ is the line $B_{i+1}'B_{i+2}'$ while the image of the line $A_{i+1}A_{i+2}$ is the circle $IA_{i+1}'A_{i+2}'$ that is tangent to $B_i'B_{i+2}'$, and $B_i'B_{i+2}'$. These three circles have equal radii, so their centers $P_1, P_2, P_3$ form a triangle also homothetic to $\triangle B_1'B_2'B_3'$. Consequently, points $A_1', A_2', A_3'$, that are the reflections of $I$ across the sides of $P_1P_2P_3$, are vertices of a triangle also homothetic to $B_1'B_2'B_3'$. It follows that $A_1'B_1', A_2'B_2', A_3'B_3'$ are concurrent at some point $J'$, i.e., that the circles $A_iB_iI$ all pass through $J$.

10. Suppose that $k \geq 4$. Consider any polynomial $F(x)$ with integer coefficients such that $0 \leq F(x) \leq k$ for $x = 0, 1, \ldots, k+1$. Since $F(k+1) - F(0)$ is divisible by $k+1$, we must have $F(k+1) = F(0)$. Hence

$$F(x) - F(0) = x(x - k - 1)Q(x)$$

for some polynomial $Q(x)$ with integer coefficients. In particular, $F(x) - F(0)$ is divisible by $x(k + 1 - x) > k + 1$ for every $x = 2, 3, \ldots, k - 1$, so $F(x) = F(0)$ must hold for any $x = 2, 3, \ldots, k - 1$. It follows that

$$F(x) - F(0) = x(x - 2)(x - 3) \cdots (x - k + 1)(x - k - 1)R(x)$$
for some polynomial $R(x)$ with integer coefficients. Thus $k \geq |F(1) - F(0)| = k(k - 2)!|R(1)|$, although $k(k - 2)! > k$ for $k \geq 4$. In this case we have $F(1) = F(0)$ and similarly $F(k) = F(0)$. Hence, the statement is true for $k \geq 4$.

It is easy to find counterexamples for $k \leq 3$. These are, for example,

$$F(x) = \begin{cases} 
  x(2-x) & \text{for } k = 1, \\
  x(3-x) & \text{for } k = 2, \\
  x(2-x)^2(4-x) & \text{for } k = 3.
\end{cases}$$

11. All real roots of $P(x)$ (if any) are negative: say $-a_1, -a_2, \ldots, -a_k$. Then $P(x)$ can be factored as

$$P(x) = C(x + a_1) \cdots (x + a_k)(x^2 - b_1x + c_1) \cdots (x^2 - b_mx + c_m),$$

where $x^2 - b_ix + c_i$ are quadratic polynomials without real roots. Since the product of polynomials with positive coefficients is again a polynomial with positive coefficients, it will be sufficient to prove the result for each of the factors in (1). The case of $x + a_j$ is trivial. It remains only to prove the claim for every polynomial $x^2 - bx + c$ with $b^2 < 4c$.

From the binomial formula, we have for any $n \in \mathbb{N}$,

$$(1 + x)^n(x^2 - bx + c) = \sum_{i=0}^{n+2} \left[ \binom{n}{i-2} - b \binom{n}{i-1} + c \binom{n}{i} \right] x^i = \sum_{i=0}^{n+2} C_i x^i,$$

where

$$C_i = \frac{n! ((b + c + 1)i^2 - ((b + 2c)n + (2b + 3c + 1))i + c(n^2 + 3n + 2))}{i!(n - i + 2)!}.$$

The coefficients $C_i$ of $x^i$ appear in the form of a quadratic polynomial in $i$ depending on $n$. We claim that for large enough $n$ this polynomial has negative discriminant, and is thus positive for every $i$. Indeed, this discriminant equals $D = ((b + 2c)n + (2b + 3c + 1))^2 - 4(b + c + 1)c(n^2 + 3n + 2) = (b^2 - 4c)n^2 - 2Un + V$, where $U = 2b^2 + bc + b - 4c$ and $V = (2b + c + 1)^2 - 4c$, and since $b^2 - 4c < 0$, for large $n$ it clearly holds that $D < 0$.

12. **Lemma.** For any polynomial $P$ of degree at most $n$, the following equality holds:

$$\sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} P(i) = 0.$$

**Proof.** See (SL81-13).

Suppose to the contrary that the degree of $f$ is at most $p - 2$. Then it follows from the lemma that

$$0 = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} f(i) \equiv \sum_{i=0}^{p-1} f(i) \pmod{p},$$
since \( \binom{p-1}{i} = \frac{(p-1)(p-2)\cdots(p-i)}{i!} \equiv (-1)^i \pmod{p} \). But this is clearly impossible if \( f(i) \) equals 0 or 1 modulo \( p \) and \( f(0) = 0, f(1) = 1 \).

**Remark.** In proving the essential relation \( \sum_{i=0}^{p-1} f(i) \equiv 0 \pmod{p} \), it is clearly enough to show that \( S_k = 1 + 2^k + \cdots + (p-1)^k \) is divisible by \( p \) for every \( k \leq p-2 \). This can be shown in two other ways.

(1) By induction. Assume that \( S_0 \equiv \cdots \equiv S_{k-1} \pmod{p} \). By the binomial formula we have

\[
0 \equiv \sum_{n=0}^{p-1} [(n+1)^{k+1} - n^{k+1}] \equiv (k+1)S_k + \sum_{i=0}^{k-1} \binom{k+1}{i} S_i \pmod{p},
\]

and the inductive step follows.

(2) Using the primitive root \( g \) modulo \( p \). Then

\[
S_k \equiv 1 + g^k + \cdots + g^{k(p-2)} = \frac{g^{k(p-1)} - 1}{g^k - 1} \equiv 0 \pmod{p}.
\]

13. Denote \( A(r) \) and \( B(r) \) by \( A(n, r) \) and \( B(n, r) \) respectively.

The numbers \( A(n, r) \) can be found directly: one can choose \( r \) girls and \( r \) boys in \( \binom{n}{r} \) ways, and pair them in \( r! \) ways. Hence

\[
A(n, r) = \binom{n}{r}^2 \cdot r! = \frac{n!^2}{(n-r)!2!}.
\]

Now we establish a recurrence relation between the \( B(n, r) \)'s. Let \( n \geq 2 \) and \( 2 \leq r \leq n \). There are two cases for a desired selection of \( r \) pairs of girls and boys:

(i) One of the girls dancing is \( g_n \). Then the other \( r-1 \) girls can choose their partners in \( B(n-1, r-1) \) ways and \( g_n \) can choose any of the remaining \( 2n-r \) boys. Thus, the total number of choices in this case is \( (2n-r)B(n-1, r-1) \).

(ii) \( g_n \) is not dancing. Then there are exactly \( B(n-1, r) \) possible choices. Therefore, for every \( n \geq 2 \) it holds that

\[
B(n, r) = (2n-r)B(n-1, r-1) + B(n-1, r) \quad \text{for } r = 2, \ldots, n.
\]

Here we assume that \( B(n, r) = 0 \) for \( r > n \), while \( B(n, 1) = 1 + 3 + \cdots + (2n-1) = n^2 \).

It is directly verified that the numbers \( A(n, r) \) satisfy the same initial conditions and recurrence relations, from which it follows that \( A(n, r) = B(n, r) \) for all \( n \) and \( r \leq n \).

14. We use the following nonstandard notation: (1°) for \( x, y \in \mathbb{N}, x \sim y \) means that \( x \) and \( y \) have the same prime divisors; (2°) for a prime \( p \) and integers \( r \geq 0 \) and \( x > 0, p^r \parallel x \) means that \( x \) is divisible by \( p^r \), but not by \( p^{r+1} \). First, \( b^m - 1 \sim b^n - 1 \) is obviously equivalent to \( b^m - 1 \sim \gcd(b^m - 1, b^n - 1) = b^d - 1 \), where \( d = \gcd(m, n) \). Setting \( b^d = a \) and \( m = kd \), we reduce
the condition of the problem to \( a^k - 1 \sim a - 1 \). We are going to show that this implies that \( a + 1 \) is a power of 2. This will imply that \( d \) is odd (for even \( d, a + 1 = b^d + 1 \) cannot be divisible by 4), and consequently \( b + 1 \), as a divisor of \( a + 1 \), is also a power of 2. But before that, we need the following important lemma (Theorem 2.126).

**Lemma.** Let \( a, k \) be positive integers and \( p \) an odd prime. If \( \alpha \geq 1 \) and \( \beta \geq 0 \) are such that \( p^\alpha \| a - 1 \) and \( p^\beta \| k \), then \( p^{\alpha+\beta} \| a^k - 1 \).

**Proof.** We use induction on \( \beta \). If \( \beta = 0 \), then \( a^k \equiv a^1 \equiv a \) (mod \( p \)) (because \( a \equiv 1 \)), and it is not divisible by \( p \).

Suppose that the lemma is true for some \( \beta \geq 0 \), and let \( k = p^{\beta+1}t \) where \( p \nmid t \). By the induction hypothesis, \( a^{k/p} = ap^\beta t = mp^{\alpha+\beta} + 1 \) for some \( m \) not divisible by \( p \). Furthermore,

\[
a^k - 1 = (mp^{\alpha+\beta} + 1)^p - 1 = (mp^{\alpha+\beta})^p + \cdots + \left(\begin{array}{c} p \\ 2 \end{array}\right)(mp^{\alpha+\beta})^2 + mp^{\alpha+\beta} + 1.
\]

Since \( p \mid \left(\begin{array}{c} p \\ 2 \end{array}\right) = \frac{p(p-1)}{2} \), all summands except for the last one are divisible by \( p^{\alpha+\beta+2} \). Hence \( p^{\alpha+\beta+1} \| a^k - 1 \), completing the induction.

Now let \( a^k - 1 \sim a - 1 \) for some \( a, k > 1 \). Suppose that \( p \) is an odd prime divisor of \( k \), with \( p^\beta \| k \). Then putting \( X = a^{p^\beta} - 1 \sim a - 1 \); hence each prime divisor \( q \) of \( X \) must also divide \( a - 1 \). But then \( a^i \equiv 1 \) (mod \( q \)) for each \( i \in \mathbb{N}_0 \), which gives us \( X \equiv p^\beta \) (mod \( q \)). Therefore \( q \mid p^\beta \), i.e., \( q = p \); hence \( X \) is a power of \( p \).

On the other hand, since \( p \mid a - 1 \), we put \( p^\alpha \| a - 1 \). From the lemma we obtain \( p^{\alpha+\beta} \| a^{p^\beta - 1} \), and deduce that \( p^\beta \| X \). But \( X \) has no prime divisors other than \( p \), so we must have \( X = p^\beta \). This is clearly impossible, because \( X > p^\beta \) for \( a > 1 \). Thus our assumption that \( k \) has an odd prime divisor leads to a contradiction: in other words, \( k \) must be a power of 2.

Now \( a^k - 1 \sim a - 1 \) implies \( a - 1 \sim a^2 - 1 = (a - 1)(a + 1) \), and thus every prime divisor \( q \) of \( a + 1 \) must also divide \( a - 1 \). Consequently \( q = 2 \), so it follows that \( a + 1 \) is a power of 2. As we explained above, this gives that \( b + 1 \) is also a power of 2.

**Remark.** In fact, one can continue and show that \( k \) must be equal to 2. It is not possible for \( a^4 - 1 \sim a^2 - 1 \) to hold. Similarly, we must have \( d = 1 \). Therefore all possible triples \((b, m, n)\) with \( m > n \) are \((2^6 - 1, 2, 1)\).

15. Let \( a + bt, t = 0, 1, 2, \ldots \), be a given arithmetic progression that contains a square and a cube \((a, b > 0)\). We use induction on the progression step \( b \) to prove that the progression contains a sixth power.

(i) \( b = 1 \): this case is trivial.

(ii) \( b = p^m \) for some prime \( p \) and \( m > 0 \). The case \( p^m \mid a \) trivially reduces to the previous case, so let us have \( p^m \nmid a \).

Suppose that \( \gcd(a, p) = 1 \). If \( x, y \) are integers such that \( x^2 \equiv y^3 \equiv a \) (here all the congruences will be mod \( p^m \)), then \( x^6 \equiv a^3 \) and \( y^6 \equiv a^2 \).

Consider an integer \( y_1 \) such that \( yy_1 \equiv 1 \). It satisfies \( a^2(xy_1)^6 \equiv
\[ x^6 y^6 y_1^6 \equiv x^6 \equiv a^3, \] and consequently \((xy_1)^6 \equiv a\). Hence a sixth power exists in the progression.

If \(\gcd(a, p) > 1\), we can write \(a = p^k c\), where \(k < m\) and \(p \nmid c\). Since the arithmetic progression \(x_t = a + bt = p^k(c + p^{m-k}t)\) contains a square, \(k\) must be even; similarly, it contains a cube, so \(3 | k\). It follows that \(6 | k\). The progression \(c + p^{m-k}t\) thus also contains a square and a cube; hence by the previous case it contains a sixth power and thus \(x_t\) does also.

(iii) \(b\) is not a power of a prime, and thus can be expressed as \(b = b_1 b_2\), where \(b_1, b_2 > 1\) and \(\gcd(b_1, b_2) = 1\). It is given that progressions \(a + bt\) and \(a + b_2t\) both contain a square and a cube, and therefore by the inductive hypothesis they both contain sixth powers: say \(z_1^6\) and \(z_2^6\), respectively. By the Chinese remainder theorem, there exists \(z \in \mathbb{N}\) such that \(z \equiv z_1 \pmod{b_1}\) and \(z \equiv z_2 \pmod{b_2}\). But then \(z^6\) belongs to both of the progressions \(a + b_1t\) and \(a + b_2t\). Hence \(z^6\) is a member of the progression \(a + bt\).

16. Let \(d_a(X), d_b(X), d_c(X)\) denote the distances of a point \(X\) interior to \(\triangle ABC\) from the lines \(BC, CA, AB\) respectively. We claim that \(X \in PQ\) if and only if \(d_a(X) + d_b(X) = d_c(X)\). Indeed, if \(X \in PQ\) and \(PX = kPQ\) then \(d_a(X) = kd_a(Q)\), \(d_b(X) = (1-k)d_b(P)\), and \(d_c(X) = (1-k)d_c(Q)\), and simple substitution yields \(d_a(X) + d_b(X) = d_c(X)\). The converse follows easily. In particular, \(O \in PQ\) if and only if \(d_a(O) + d_b(O) = d_c(O)\), i.e., \(\cos \alpha + \cos \beta = \cos \gamma\).

We shall now show that \(I \in DE\) if and only if \(AE + BD = DE\). Let \(K\) be the point on the segment \(DE\) such that \(AE = EK\). Then \(\angle EKA = \frac{1}{2} \angle DEC = \frac{1}{2} \angle CBA = \angle IBA\); hence the points \(A, B, I, K\) are concyclic.

The point \(I\) lies on \(DE\) if and only if \(\angle BKD = \angle BAI = \frac{1}{2} \angle BAC = \frac{1}{2} \angle CDE = \angle DBK\), which is equivalent to \(KD = BD\), i.e., to \(AE + BD = DE\). But since \(AE = AB \cos \alpha, BD = AB \cos \beta,\) and \(DE = AB \cos \gamma\), we have that \(I \in DE \iff \cos \alpha + \cos \beta = \cos \gamma\). The conditions for \(O \in PQ\) and \(I \in DE\) are thus equivalent.

Second solution. We know that three points \(X, Y, Z\) are collinear if and only if for some \(\lambda, \mu \in \mathbb{R}\) with sum 1, we have \(\lambda CX + \mu CY = CZ\).

Specially, if \(\overrightarrow{CX} = p \overrightarrow{CA}\) and \(\overrightarrow{CY} = q \overrightarrow{CB}\) for some \(p, q\), and \(\overrightarrow{CZ} = k \overrightarrow{CA} + l \overrightarrow{CB}\), then \(Z\) lies on \(XY\) if and only if \(kq + lp = pq\).

Using known relations in a triangle we directly obtain

\[
\overrightarrow{CP} = \frac{\sin \beta}{\sin \beta + \sin \gamma} \overrightarrow{CB}, \quad \overrightarrow{CQ} = \frac{\sin \alpha}{\sin \alpha + \sin \gamma} \overrightarrow{CA},
\]

\[
\overrightarrow{CO} = \frac{\sin 2\alpha \cdot CA + \sin 2\beta \cdot CB}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma}; \quad \overrightarrow{CD} = \frac{\tan \beta}{\tan \beta + \tan \gamma} \overrightarrow{CB},
\]

\[
\overrightarrow{CE} = \frac{\tan \beta}{\tan \beta + \tan \gamma} \overrightarrow{CA}, \quad \overrightarrow{CI} = \frac{\sin \alpha \cdot CA + \sin \beta \cdot CB}{\sin \alpha + \sin \beta + \sin \gamma}.
\]
Now by the above considerations we get that the conditions (1) $P, Q, O$ are collinear and (2) $D, E, I$ are collinear are both equivalent to $\cos \alpha + \cos \beta = \cos \gamma$.

17. We note first that $x$ and $y$ must be powers of the same positive integer. Indeed, if $x = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $y = p_1^{\beta_1} \cdots p_k^{\beta_k}$ (some of the $\alpha_i$ and $\beta_i$ may be 0, but not both for the same index $i$), then $x^{q^2} = y^x$ implies $\frac{\alpha_i}{\beta_i} = \frac{x^q}{y^x} = \frac{p}{q}$ for some $p, q > 0$ with $\gcd(p, q) = 1$, so for $a = p_1^{\alpha_1/p} \cdots p_k^{\alpha_k/p}$ we can take $x = a^p$ and $y = a^q$.

If $a = 1$, then $(x, y) = (1, 1)$ is the trivial solution. Let $a > 1$. The given equation becomes $a^{pa^{2q}} = a^{qa^p}$, which reduces to $pa^{2q} = qa^p$. Hence $p \neq q$, so we distinguish two cases:

(i) $p > q$. Then from $a^{2d} < a^p$ we deduce $p > 2q$. We can rewrite the equation as $p = a^{p-2q}q$, and putting $p = 2q + d$, $d > 0$, we obtain $d = q(a^d - 2)$. By induction, $2d - 2 > d$ for each $d > 2$, so we must have $d \leq 2$. For $d = 1$ we get $q = 1$ and $a = p = 3$, and therefore $(x, y) = (27, 3)$, which is indeed a solution. For $d = 2$ we get $q = 1$, $a = 2$, and $p = 4$, so $(x, y) = (16, 2)$, which is another solution.

(ii) $p < q$. As above, we get $q/p = a^{2q-p}$, and setting $d = 2q - p > 0$, this is transformed to $a^d = a^{(2a^d-1)p}$, or equivalently to $d = (2a^d - 1)p$.

However, this equality cannot hold, because $2a^d - 1 > d$ for each $a \geq 2$, $d \geq 1$.

The only solutions are thus $(1, 1), (16, 2)$, and $(27, 3)$.

18. By symmetry, assume that $AB > AC$. The point $D$ lies between $M$ and $P$ as well as between $Q$ and $R$, and if we show that $DM \cdot DP = DQ \cdot DR$, it will imply that $M, P, Q, R$ lie on a circle.

Since the triangles $ABC, AEF, AQR$ are similar, the points $B, C, Q, R$ lie on a circle. Hence $DB \cdot DC = DQ \cdot DR$, and it remains to prove that

$$DB \cdot DC = DM \cdot DP.$$  

However, the points $B, C, E, F$ are concyclic, but so are the points $E, F, D, M$ (they lie on the nine-point circle), and we obtain $PB \cdot PC = PE \cdot PF = PD \cdot PM$. Set $PB = x$ and $PC = y$. We have $PM = \frac{x^2 + y^2}{2}$ and hence $PD = \frac{2xy}{x+y}$. It follows that $DB = PB - PD = \frac{x(x-y)}{x+y}$, $DC = \frac{y(x-y)}{x+y}$, and $DM = \frac{(x-y)^2}{2(x+y)}$, from which we immediately obtain

$$DB \cdot DC = DM \cdot DP = \frac{xy(x-y)^2}{(x+y)^2},$$ as needed.

19. Using that $a_{n+1} = 0$ we can transform the desired inequality into

$$\sqrt{a_1 + a_2 + \cdots + a_{n+1}} \leq \sqrt{1} \sqrt{a_1} + (\sqrt{2} - \sqrt{1}) \sqrt{a_2} + \cdots + (\sqrt{n+1} - \sqrt{n}) \sqrt{a_{n+1}}.$$  

We shall prove by induction on $n$ that (1) holds for any $a_1 \geq a_2 \geq \cdots \geq a_{n+1} \geq 0$, i.e., not only when $a_{n+1} = 0$. For $n = 0$ the inequality is
obvious. For the inductive step from \( n - 1 \) to \( n \), where \( n \geq 1 \), we need to prove the inequality
\[
\sqrt{a_1 + \cdots + a_{n+1}} - \sqrt{a_1 + \cdots + a_n} \leq \left( \sqrt{n+1} - \sqrt{n} \right) \sqrt{a_{n+1}}. \tag{2}
\]
Putting \( S = a_1 + a_2 + \cdots + a_n \), this simplifies to \( \sqrt{S + a_{n+1}} - \sqrt{S} \leq \sqrt{na_{n+1} + a_{n+1}} - \sqrt{na_{n+1}} \). For \( a_{n+1} = 0 \) the inequality is obvious. For \( a_{n+1} > 0 \) we have that the function \( f(x) = \sqrt{x + a_{n+1}} - \sqrt{x} = \frac{a_{n+1}}{\sqrt{x^2 + a_{n+1} + \sqrt{x}}} \) is strictly decreasing on \( \mathbb{R}^+ \); hence (2) will follow if we show that \( S \geq na_{n+1} \). However, this last is true because \( a_1, \ldots, a_n \geq a_{n+1} \).

Equality holds if and only if \( a_1 = a_2 = \cdots = a_k \) and \( a_k+1 = \cdots = a_{n+1} = 0 \) for some \( k \).

**Second solution.** Setting \( b_k = \sqrt{a_k} - \sqrt{a_{k+1}} \) for \( k = 1, \ldots, n \) we have \( a_i = (b_i + \cdots + b_n)^2 \), so the desired inequality after squaring becomes
\[
\sum_{k=1}^{n} kb_k^2 + 2 \sum_{1 \leq k < t \leq n} kb_kb_t \leq \sum_{k=1}^{n} k b_k^2 + 2 \sum_{1 \leq k < t \leq n} \sqrt{kt} b_kb_t,
\]
which clearly holds.

20. To avoid dividing into cases regarding the position of the point \( X \), we use oriented angles.

Let \( R \) be the foot of the perpendicular from \( X \) to \( BC \). It is well known that the points \( P, Q, R \) lie on the corresponding Simson line. This line is a tangent to \( \gamma \) (i.e., the circle \( XDR \)) if and only if \( \angle PRD = \angle RXD \). We have
\[
\angle PRD = \angle PXB = 90° - \angle XBA = 90° - \angle XBC + \angle ABC
\]
\[
= 90° - \angle DAC + \angle ABC
\]
and
\[
\angle RXD = 90° - \angle ADB = 90° + \angle BCA - \angle DAC;
\]
hence \( \angle PRD = \angle RXD \) if and only if \( \angle ABC = \angle BCA \), i.e, \( AB = AC \).

21. For any permutation \( \pi = (y_1, y_2, \ldots, y_n) \) of \( (x_1, x_2, \ldots, x_n) \), denote by \( S(\pi) \) the sum \( y_1 + 2y_2 + \cdots + ny_n \). Suppose, contrary to the claim, that \( |S(\pi)| > \frac{n+1}{2} \) for any \( \pi \).

Further, we note that if \( \pi' \) is obtained from \( \pi \) by interchanging two neighboring elements, say \( y_k \) and \( y_{k+1} \), then \( S(\pi) \) and \( S(\pi') \) differ by \( |y_k + y_{k+1}| \leq n + 1 \), and consequently they must be of the same sign.

Now consider the identity permutation \( \pi_0 = (x_1, \ldots, x_n) \) and the reverse permutation \( \pi_n = (x_n, \ldots, x_1) \). There is a sequence of permutations \( \pi_0, \pi_1, \ldots, \pi_m = \pi_0 \) such that for each \( i \), \( \pi_{i+1} \) is obtained from \( \pi_i \) by interchanging two neighboring elements. Indeed, by successive interchanges we can put \( x_n \) in the first place, then \( x_{n-1} \) in the second place, etc. Hence all \( S(\pi_0), \ldots, S(\pi_m) \) are of the same sign. However, since \( |S(\pi_0) + S(\pi_m)| = (n + 1)|x_1 + \cdots + x_n| = n + 1 \), this implies that one of
22. (a) Suppose that \( f \) and \( g \) are such functions. From \( g(f(x)) = x^3 \) we have \( f(x_1) \neq f(x_2) \) whenever \( x_1 \neq x_2 \). In particular, \( f(-1), f(0), \) and \( f(1) \) are three distinct numbers. However, since \( f(x)^2 = f(g(f(x))) = f(x^3) \), each of the numbers \( f(-1), f(0), f(1) \) is equal to its square, and so must be either 0 or 1. This contradiction shows that no such \( f, g \) exist.

(b) The answer is yes. We begin with constructing functions \( F, G : (1, \infty) \to (1, \infty) \) with the property \( F(G(x)) = x^2 \) and \( G(F(x)) = x^4 \) for \( x > 1 \). Define the functions \( \varphi, \psi \) by \( F(2^t) = 2^{2^t} \) and \( G(2^t) = 2^{2^\psi(t)} \). These functions determine \( F \) and \( G \) on the entire interval \((1, \infty)\), and satisfy \( \varphi(\psi(t)) = t + 1 \) and \( \psi(\varphi(t)) = t + 2 \). It is easy to find examples of \( \varphi \) and \( \psi \): for example, \( \varphi(t) = \frac{1}{2} t + 1, \psi(t) = 2t \). Thus we also arrive at an example for \( F, G \):

\[
F(x) = 2^{2^{\frac{1}{2} \log_2 \log_2 x + 1}} = 2^{\sqrt{\log_2 x}}, \quad G(x) = 2^{2^{2 \log_2 \log_2 x}} = 2^{\log_2^2 x}.
\]

It remains only to extend these functions to the whole of \( \mathbb{R} \). This can be done as follows:

\[
\tilde{f}(x) = \begin{cases} 
F(x) & \text{for } x > 1, \\
1/F(1/x) & \text{for } 0 < x < 1, \\
x & \text{for } x \in \{0, 1\};
\end{cases} 
\quad \tilde{g}(x) = \begin{cases} 
G(x) & \text{for } x > 1, \\
1/G(1/x) & \text{for } 0 < x < 1, \\
x & \text{for } x \in \{0, 1\};
\end{cases}
\]

and then \( f(x) = \tilde{f}(|x|) \), \( g(x) = \tilde{g}(|x|) \) for \( x \in \mathbb{R} \).

It is directly verified that these functions have the required property.

23. Let \( K, L, M, \) and \( N \) be the projections of \( O \) onto the lines \( AB, BC, CD, \) and \( DA \), and let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \) denote the angles \( OAB, OBC, OCD, ODA, OAD, OBA, OCB, ODC \), respectively.

We start with the following observation: Since \( NK \) is a chord of the circle with diameter \( OA \), we have \( OA \sin \angle A = NK = ON \cos \alpha_1 + OK \cos \beta_1 \) (because \( \angle ONK = \alpha_1 \) and \( \angle OKN = \beta_1 \)). Analogous equalities also hold: \( OB \sin \angle B = KL = OK \cos \alpha_2 + OL \cos \beta_2 \), \( OC \sin \angle C = LM = OL \cos \alpha_3 + OM \cos \beta_3 \) and \( OD \sin \angle D = MN = OM \cos \alpha_4 + ON \cos \beta_4 \). Now the condition in the problem can be restated as \( NK + LM = KL + MN \) (i.e., \( KLMN \) is circumscribed), i.e.,

\[
OK(\cos \beta_1 - \cos \alpha_2) + OL(\cos \alpha_3 - \cos \beta_2) + OM(\cos \beta_3 - \cos \alpha_4) + ON(\cos \alpha_1 - \cos \beta_4) = 0. \tag{1}
\]

To prove that \( ABCD \) is cyclic, it suffices to show that \( \alpha_1 = \beta_1 \). Assume the contrary, and let w.l.o.g. \( \alpha_1 > \beta_1 \). Then point \( A \) lies inside the circle \( BCD \), which is further equivalent to \( \beta_1 > \alpha_2 \). On the other hand, from \( \alpha_1 + \beta_2 = \alpha_3 + \beta_4 \) we deduce \( \alpha_3 > \beta_2 \), and similarly \( \beta_3 > \alpha_4 \). Therefore,
since the cosine is strictly decreasing on \((0, \pi)\), the left side of \((1)\) is strictly negative, yielding a contradiction.

24. There is a bijective correspondence between representations in the given form of \(2k\) and \(2k + 1\) for \(k = 0, 1, \ldots\), since adding 1 to every representation of \(2k\), we obtain a representation of \(2k + 1\), and conversely, every representation of \(2k + 1\) contains at least one 1, which can be removed. Hence, \(f(2k + 1) = f(2k)\).

Consider all representations of \(2k\). The number of those that contain at least one 1 equals \(f(2k - 1) = f(2k - 2)\), while the number of those not containing a 1 equals \(f(k)\) (the correspondence is given by division of summands by 2). Therefore

\[
f(2k) = f(2k - 2) + f(k). \tag{1}
\]

Summing these equalities over \(k = 1, \ldots, n\), we obtain

\[
f(2n) = f(0) + f(1) + \cdots + f(n). \tag{2}
\]

We first prove the right-hand inequality. Since \(f\) is increasing, and \(f(0) + f(1) = f(2)\), \(\text{(2)}\) yields \(f(2n) \leq nf(n)\) for \(n \geq 2\). Now \(f(2^3) = f(0) + \cdots + f(4) = 10 < 2^{2^2/2}\), and one can easily conclude by induction that \(f(2n+1) \leq 2nf(2^n) \leq 2^n \cdot 2^{n^2/2} < 2^{(n+1)^2/2}\) for each \(n \geq 3\).

We now derive the lower estimate. It follows from \((1)\) that \(f(x + 2) - f(x)\) is increasing. Consequently, for each \(m\) and \(k < m\) we have \(f(2m + 2k) - f(2m) \geq f(2m + 2k - 2) - f(2m - 2) \geq \cdots \geq f(2m) - f(2m - 2k)\), so \(f(2m + 2k) + f(2m - 2k) \geq 2f(2m)\). Adding all these inequalities for \(k = 1, 2, \ldots, m\), we obtain \(f(0) + f(2) + \cdots + f(4m) \geq (2m + 1)f(2m)\). But since \(f(2) = f(3)\), \(f(4) = f(5)\) etc., we also have \(f(1) + f(3) + \cdots + f(4m - 1) > (2m - 1)f(2m)\), which together with the above inequality gives

\[
f(8m) = f(0) + f(1) + \cdots + f(4m) > 4mf(2m). \tag{3}
\]

Finally, we have that the inequality \(f(2^n) > 2^{n^2/4}\) holds for \(n = 2\) and \(n = 3\), while for larger \(n\) we have by induction \(f(2^n) > 2^{n-1} f(2^{n-2}) > 2^{n-1+(n-2)^2/4} = 2^{n^2/4}\). This completes the proof.

**Remark.** Despite the fact that the lower estimate is more difficult, it is much weaker than the upper estimate. It can be shown that \(f(2^n)\) eventually (for large \(n\)) exceeds \(2^{cn^2}\) for any \(c < 1/2\).

25. Let \(MR\) meet the circumcircle of triangle \(ABC\) again at a point \(X\). We claim that \(X\) is the common point of the lines \(KP, LQ, MR\). By symmetry, it will be enough to show that \(X\) lies on \(KP\). It is easy to see that \(X\) and \(P\) lie on the same side of \(AB\) as \(K\). Let \(I_a = AK \cap BP\) be the excenter of \(\triangle ABC\) corresponding to \(A\). It is easy to calculate that \(\angle AI_aB = \gamma/2\), from which we get \(\angle RPB = \angle AI_aB = \angle MCB = \angle RXB\). Therefore \(R, B, P, X\) are concyclic. Now if \(P\) and \(K\) are on distinct sides of \(BX\) (the
other case is similar), we have \[
\angle RXP = 180^\circ - \angle RBP = 90^\circ - \beta/2 = \angle MAK = 180^\circ - \angle RXK,
\]
from which it follows that \(K, X, P\) are collinear, as claimed.

**Remark.** It is not essential for the statement of the problem that \(R\) be an internal point of \(AB\). Work with cases can be avoided using oriented angles.

26. Let us first examine the case that all the inequalities in the problem are actually equalities. Then \(a_{n-2} = a_{n-1} + a_n, a_{n-3} = 2a_{n-1} + a_n, \ldots, a_0 = F_n a_{n-1} + F_{n-1} a_n = 1,\) where \(F_n\) is the \(n\)th Fibonacci number. Then it is easy to see (from \(F_1 + F_2 + \cdots + F_k = F_{k+2}\)) that \(a_0 + \cdots + a_n = (F_{n+2} - 1)a_{n-1} + F_{n+1} a_n = \frac{F_{n+2} - 1}{F_n} a_n + \left(\frac{F_{n+1} - F_n (F_{n+2} - 1)}{F_n^2}\right) a_n.\) Since \(\frac{F_n - 1}{F_n^2} \leq F_{n+1},\) it follows that \(a_0 + a_1 + \cdots + a_n \geq \frac{F_{n+2} - 1}{F_n^2} a_n,\) with equality holding if and only if \(a_n = 0\) and \(a_{n-1} = \frac{1}{F_n} a_n.\)

We denote by \(M_n\) the required minimum in the general case. We shall prove by induction that \(M_n = \frac{F_{n+2} - 1}{F_n^2}.\) For \(M_1 = 1\) and \(M_2 = 2\) it is easy to show that the formula holds; hence the inductive basis is true. Suppose that \(n > 2.\) The sequences \(1, \frac{a_2}{a_1}, \ldots, \frac{a_n}{a_1}\) and \(1, \frac{a_3}{a_2}, \ldots, \frac{a_n}{a_2}\) also satisfy the conditions of the problem. Hence we have

\[
a_0 + \cdots + a_n = a_0 + a_1 \left(1 + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_1}\right) \geq 1 + a_1 M_{n-1}
\]
and

\[
a_0 + \cdots + a_n = a_0 + a_1 + a_2 \left(1 + \frac{a_3}{a_2} + \cdots + \frac{a_n}{a_2}\right) \geq 1 + a_1 + a_2 M_{n-2}.
\]

Multiplying the first inequality by \(M_{n-2} - 1\) and the second one by \(M_{n-1},\) adding the inequalities and using that \(a_1 + a_2 \geq 1,\) we obtain \((M_{n-1} + M_{n-2} + 1)(a_0 + \cdots + a_n) \geq M_{n-1} M_{n-2} + M_{n-1} + M_{n-2} + 1,\) so

\[
M_n \geq \frac{M_{n-1} M_{n-2} + M_{n-1} + M_{n-2} + 1}{M_{n-1} + M_{n-2} + 1}.
\]

Since \(M_{n-1} = \frac{F_{n+1} - 1}{F_{n-1}}\) and \(M_{n-2} = \frac{F_n - 1}{F_{n-2}}\), the above inequality easily yields \(M_n \geq \frac{F_{n+2} - 1}{F_n^2}.\) However, we have shown above that equality can occur; hence \(\frac{F_{n+2} - 1}{F_n^2}\) is indeed the required minimum.
4.39 Solutions to the Shortlisted Problems of IMO 1998

1. We begin with the following observation: Suppose that $P$ lies in $\triangle AEB$, where $E$ is the intersection of $AC$ and $BD$ (the other cases are similar). Let $M, N$ be the feet of the perpendiculars from $P$ to $AC$ and $BD$ respectively. We have $S_{ABP} = S_{ABE} - S_{AEP} - S_{BEP} = \frac{1}{2}(AE \cdot BE - AE \cdot EN - BE \cdot EM) = \frac{1}{2}(AM \cdot BN - EM \cdot EN)$. Similarly, $S_{CDP} = \frac{1}{2}(CN \cdot DP - EM \cdot EN)$. Therefore, we obtain

$$S_{ABP} - S_{CDP} = \frac{1}{4}(AM \cdot BN - CM \cdot DN).$$

Now suppose that $ABCD$ is cyclic. Then $P$ is the circumcenter of $ABCD$; hence $M$ and $N$ are the midpoints of $AC$ and $BD$. Hence $AM = CM$ and $BN = DN$; thus (1) gives us $S_{ABP} = S_{CDP}$.

On the other hand, suppose that $ABCD$ is not cyclic and let w.l.o.g. $PA = PB > PC = PD$. Then we must have $AM > CM$ and $BN > DN$, and consequently by (1), $S_{ABP} > S_{CDP}$. This proves the other implication.

Second solution. Let $F$ and $G$ denote the midpoints of $AB$ and $CD$, and assume that $P$ is on the same side of $FG$ as $B$ and $C$. Since $PF \perp AB$, $PG \perp CD$, and $\angle FEB = \angle ABE$, $\angle GEC = \angle DCE$, a direct computation yields $\angle FPG = \angle FEG = 90^\circ + \angle ABE + \angle DCE$.

Taking into account that $S_{ABP} = \frac{1}{2}AB \cdot FP = FE \cdot FP$, we note that $S_{ABP} = S_{CDP}$ is equivalent to $FE \cdot FP = GE \cdot GP$, i.e., to $FE/EG = GP/PF$. But this last is equivalent to triangles $EFG$ and $PGF$ being similar, which holds if and only if $EFPG$ is a parallelogram. This last is equivalent to $\angle EFP = \angle EGP$, or $2\angle ABE = 2\angle DCE$. Thus $S_{ABP} = S_{CDP}$ is equivalent to $ABCD$ being cyclic.

Remark. The problems also allows an analytic solution, for example putting the $x$ and $y$ axes along the diagonals $AC$ and $BD$.

2. If $AD$ and $BC$ are parallel, then $ABCD$ is an isosceles trapezoid with $AB = CD$, so $P$ is the midpoint of $EF$. Let $M$ and $N$ be the midpoints of $AB$ and $CD$. Then $MN \parallel BC$, and the distance $d(E, MN)$ equals the distance $d(F, MN)$ because $B$ and $D$ are the same distance from $MN$ and $EM/BM = FN/DN$. It follows that the midpoint $P$ of $EF$ lies on $MN$, and consequently $S_{APD} : S_{BPC} = AD : BC$.

If $AD$ and $BC$ are not parallel, then they meet at some point $Q$. It is plain that $\triangle QAB \sim \triangle QCD$, and since $AE/AB = CF/CD$, we also deduce that $\triangle QAE \sim \triangle QCF$. Therefore $\angle AQE = \angle CQF$. Further, from these similarities we obtain $QE/QF = QA/QC = AB/CD = PE/PF$, etc.
which in turn means that $QP$ is the internal bisector of $\angle EQF$. But since $\angle AQE = \angle CQF$, this is also the internal bisector of $\angle AQB$. Hence $P$ is at equal distances from $AD$ and $BC$, so again $S_{APD} : S_{BPC} = AD : BC$.

**Remark.** The part $AB \parallel CD$ could also be regarded as a limiting case of the other part.

**Second solution.** Denote $\lambda = \frac{AE}{AB}$, $AB = a$, $BC = b$, $CD = c$, $DA = d$, $\angle DAB = \alpha$, $\angle ABC = \beta$. Since $d(P, AD) = \frac{c \cdot d(E, AD) + a \cdot d(F, AD)}{a + c}$, we have $S_{APD} = \frac{cS_{EAD} + aS_{FAD}}{a + c} = \frac{\lambda cS_{ABD} + (1 - \lambda) aS_{ACD}}{a + c}$. Since $S_{ABD} = \frac{1}{2} ab \sin \alpha$ and $S_{ACD} = \frac{1}{2} cd \sin \beta$, we are led to $S_{APD} = \frac{acd}{a + c} [\lambda \sin \alpha + (1 - \lambda) \sin \beta]$, and analogously $S_{BPC} = \frac{abc}{a + c} [\lambda \sin \alpha + (1 - \lambda) \sin \beta]$. Thus we obtain $S_{APD} : S_{BPC} = d : b$.

3. **Lemma.** If $U, W, V$ are three points on a line $l$ in this order, and $X$ a point in the plane with $XW \perp UV$, then $\angle UXV < 90^\circ$ if and only if $XW^2 > UW \cdot VW$.

**Proof.** Let $XW^2 > UW \cdot VW$, and let $X_0$ be a point on the segment $XW$ such that $X_0W^2 \geq UW \cdot VW$. Then $X_0W/UW = VW/X_0W$, so that triangles $X_0WU$ and $VWX_0$ are similar. Thus $\angle UX_0V = \angle UX_0W + \angle WUX_0 = 90^\circ$, which immediately implies that $\angle UXV < 90^\circ$.

Similarly, if $XW^2 \leq UW \cdot VW$, then $\angle UXV \geq 90^\circ$.

Since $BI \perp RS$, it will be enough by the lemma to show that $BI^2 > BR \cdot BS$. Note that $\triangle BKR \sim \triangle BSL$: in fact, we have $\angle KBR = \angle SLB = 90^\circ - \beta/2$ and $\angle BKR = \angle AKM = \angle KLM = \angle BSL = 90^\circ - \alpha/2$. In particular, we obtain $BR/BK = BL/BS = BK/BS$, so that $BR \cdot BS = BK^2 < BI^2$.

**Second solution.** Let $E, F$ be the midpoints of $KM$ and $LM$ respectively. The quadrilaterals $RBIE$ and $SBIF$ are inscribed in the circles with diameters $IR$ and $IS$. Now we have $\angle RIS = \angle RMS + \angleIRM + \angle ISM = 90^\circ - \beta/2 + \angle IBE + \angle IFB = 90^\circ - \beta/2 + \angle EBF$.

On the other hand, $BE$ and $BF$ are medians in $\triangle BKM$ and $\triangle BLM$ in which $BM > BK$ and $BM > BL$. We conclude that $\angle MBE < \frac{1}{2} \angle MBK$ and $\angle MFB < \frac{1}{2} \angle MBL$. Adding these two inequalities gives $\angle EBF < \beta/2$. Therefore $\angle RIS < 90^\circ$.

**Remark.** It can be shown (using vectors) that the statement remains true for an arbitrary line $t$ passing through $B$.

4. Let $K$ be the point on the ray $BN$ with $\angle BCK = \angle BMA$. Since $\angle KBC = \angle ABM$, we get $\triangle BCK \sim \triangle BMA$. It follows that $BC/BM = BK/BA$, which implies that also $\triangle BAK \sim \triangle BMC$. The quadrilateral $ANCK$ is cyclic, because $\angle BKC = \angle BAM = \angle NAC$. Then by Ptolemy’s theorem we obtain

$$AC \cdot BK = AC \cdot BN + AN \cdot CK + CN \cdot AK.$$  \hspace{1cm} (1)

On the other hand, from the similarities noted above we get
\[ CK = \frac{BC \cdot AM}{BM}, \quad AK = \frac{AB \cdot CM}{BM} \quad \text{and} \quad BK = \frac{AB \cdot BC}{BM}. \]

After substitution of these values, the equality (1) becomes

\[ \frac{AB \cdot BC \cdot AC}{BM} = AC \cdot BN + \frac{BC \cdot AM \cdot AN}{BM} + \frac{AB \cdot CM \cdot CN}{BM}, \]

which is exactly the equality we must prove multiplied by \( \frac{AB \cdot BC \cdot CA}{BM} \).

5. Let \( G \) be the centroid of \( \triangle ABC \) and \( H \) the homothety with center \( G \) and ratio \(-\frac{1}{2}\). It is well-known that \( H \) maps \( H \) into \( O \). For every other point \( X \), let us denote by \( X' \) its image under \( H \). Also, let \( A_2B_2C_2 \) be the triangle in which \( A, B, C \) are the midpoints of \( B_2C_2, C_2A_2, \) and \( A_2B_2 \), respectively.

It is clear that \( A', B', C' \) are the midpoints of sides \( BC, CA, AB \) respectively. Furthermore, \( D' \) is the reflection of \( A' \) across \( B'C' \). Thus \( D' \) must lie on \( B_2C_2 \) and \( A'D' \perp B_2C_2 \). However, it also holds that \( O A' \perp B_2C_2 \), so we conclude that \( O, D', A' \) are collinear and \( D' \) is the projection of \( O \) on \( B_2C_2 \). Analogously, \( E', F' \) are the projections of \( O \) on \( C_2A_2 \) and \( A_2B_2 \).

Now we apply Simson’s theorem. It claims that \( D', E', F' \) are collinear (which is equivalent to \( D, E, F \) being collinear) if and only if \( O \) lies on the circumcircle of \( A_2B_2C_2 \). However, this circumcircle is centered at \( H \) with radius \( 2R \), so the last condition is equivalent to \( HO = 2R \).

6. Let \( P \) be the point such that \( \triangle CDP \) and \( \triangle CBA \) are similar and equally oriented. Since then \( \angle DCP = \angle BCA \) and \( \frac{BC}{CA} = \frac{DC}{CP} \), it follows that \( \angle ACB = \angle BCD \) and \( \frac{AC}{CD} = \frac{BC}{CD} \), so \( \triangle ACP \sim \triangle BCD \). In particular, \( \frac{BC}{CA} = \frac{DB}{PA} \).

Furthermore, by the conditions of the problem we have \( \angle EDP = 360^\circ - \angle B - \angle D = \angle F \) and \( \frac{PD}{DE} = \frac{PD}{CD} \cdot \frac{CD}{DE} = \frac{AB}{BC} \cdot \frac{CD}{DE} = \frac{AE}{EF} \). Therefore \( \triangle EDP \sim \triangle EFA \) as well, so that similarly as above we conclude that \( \triangle AEP \sim \triangle FED \) and consequently \( \frac{AE}{EF} = \frac{PD}{DF} \).

Finally, \( \frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{ED}{DB} = \frac{DB}{PA} \cdot \frac{PA}{PD} \cdot \frac{PD}{DB} = 1 \).

Second solution. Let \( a, b, c, d, e, f \) be the complex coordinates of \( A, B, C, D, E, F \), respectively. The condition of the problem implies that \( \frac{a-b}{d-c} \cdot \frac{e-f}{f-a} - 1 \).

On the other hand, since \((a-b)(c-d)(e-f) + (b-c)(d-e)(f-a) = (b-c)(a-e)(f-d) + (c-a)(e-f)(d-b) \) holds identically, we immediately deduce that \( \frac{b-c}{c-a} \cdot \frac{a-e}{e-f} \cdot \frac{f-d}{d-b} = -1 \). Taking absolute values gives \( \frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{ED}{DB} = 1 \).
7. We shall use the following result.

Lemma. In a triangle $ABC$ with $BC = a$, $CA = b$, and $AB = c$,

i. $\angle C = 2\angle B$ if and only if $c^2 = b^2 + ab$;

ii. $\angle C + 180^\circ = 2\angle B$ if and only if $c^2 = b^2 - ab$.

Proof.

i. Take a point $D$ on the extension of $BC$ over $C$ such that $CD = b$. The condition $\angle C = 2\angle B$ is equivalent to $\angle ADC = \frac{1}{2} \angle C = \angle B$, and thus to $AD = AB = c$. This is further equivalent to triangles $CAD$ and $ABD$ being similar, so $CA/AD = AB/BD$, i.e., $c^2 = b(a + b)$.

ii. Take a point $E$ on the ray $CB$ such that $CE = b$. As above, $\angle C + 180^\circ = 2\angle B$ if and only if $\triangle CAE \sim \triangle ABE$, which is equivalent to $EB/BA = EA/AC$, or $c^2 = b(b - a)$.

Let $F, G$ be points on the ray $CB$ such that $CF = \frac{1}{3}a$ and $CG = \frac{4}{3}a$. Set $BC = a$, $CA = b$, $AB = c$, $EC = b_1$, and $EB = c_1$. By the lemma it follows that $c_1^2 = b_1^2 + ab$. Also $b_1 = AG$ and $c_1 = AF$, so Stewart’s theorem gives us $c_1^2 = \frac{2}{3}b^2 - \frac{1}{3}a^2 = b^2 + \frac{1}{3}ab - \frac{4}{3}a^2$ and $b_1 = -\frac{1}{3}b^2 + \frac{4}{3}a^2 = b_1^2 + \frac{1}{3}ab + \frac{4}{3}a^2$. It follows that $b_1 = \frac{2}{3}a + b$ and $c_1^2 = b_1^2 - (ab + \frac{4}{3}a^2) = b_1^2 - ab$. The statement of the problem follows immediately by the lemma.

8. Let $M$ be the point of intersection of $AE$ and $BC$, and let $N$ be the point on $\omega$ diametrically opposite $A$.

Since $\angle B < \angle C$, points $N$ and $B$ are on the same side of $AE$. Furthermore, $\angle NAE = \angle BAX = 90^\circ - \angle ABE$; hence the triangles $NAE$ and $BAX$ are similar. Consequently, $\triangle BAY$ and $\triangle NAM$ are also similar, since $M$ is the midpoint of $AE$. Thus $\angle ANZ = \angle ABZ = \angle ABE = \angle ANM$, implying that $N, M, Z$ are collinear. Now we have $\angle ZMD = 90^\circ - \angle ZMA = \angle EAZ = \angle ZED$ (the last equality because $ED$ is tangent to $\omega$); hence $ZMED$ is a cyclic quadrilateral. It follows that $\angle ZDM = \angle ZEA = \angle ZAD$, which is enough to conclude that $MD$ is tangent to the circumcircle of $AZD$.

Remark. The statement remains valid if $\angle B \geq \angle C$.

9. Set $a_{n+1} = 1 - (a_1 + \cdots + a_n)$. Then $a_{n+1} > 0$, and the desired inequality becomes

$$\frac{a_1a_2 \cdots a_{n+1}}{(1 - a_1)(1 - a_2) \cdots (1 - a_{n+1})} \leq \frac{1}{n^{n+1}}.$$ 

To prove it, we observe that

$$1 - a_i = a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{n+1} \geq n \sqrt[n]{a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n+1}}.$$ 

Multiplying these inequalities for $i = 1, 2, \ldots, n + 1$, we get exactly the inequality we need.
10. We shall first prove the inequality for \( n \) of the form \( 2^k \), \( k = 0, 1, 2, \ldots \).

The case \( k = 0 \) is clear. For \( k = 1 \), we have

\[
\frac{1}{r_1 + 1} + \frac{1}{r_2 + 1} - \frac{2}{\sqrt{r_1 r_2} + 1} = \frac{(\sqrt{r_1} \sqrt{r_2} - \sqrt{r_1} - \sqrt{r_2})^2}{(r_1 + 1)(r_2 + 1)(\sqrt{r_1 r_2} + 1)} \geq 0.
\]

For the inductive step it suffices to show that the claim for \( k \) and 2 implies that for \( k + 1 \). Indeed,

\[
\sum_{i=1}^{2^{k+1}} \frac{1}{r_i + 1} \geq \frac{2^k}{\sqrt{r_1 r_2 \cdots r_{2^k} + 1}} + \frac{2^k}{2^{k+1}} \geq \frac{2^k}{\sqrt{r_1 \cdots r_{2^k+1} + 1}},
\]

and the induction is complete.

We now show that if the statement holds for \( 2^k \), then it holds for every \( n < 2^k \) as well. Put \( r_{n+1} = r_{n+2} = \cdots = r_{2^k} = \sqrt{r_1 r_2 \cdots r_n} \). Then (1) becomes

\[
\frac{1}{r_1 + 1} + \cdots + \frac{1}{r_n + 1} + \frac{2^k - n}{\sqrt{r_1 \cdots r_{2^k} + 1}} \geq \frac{2^k}{\sqrt{r_1 \cdots r_{2^k+1} + 1}}.
\]

This proves the claim.

**Second solution.** Define \( r_i = e^{x_i} \), where \( x_i > 0 \). The function \( f(x) = \frac{1}{1 + e^x} \)
is convex for \( x > 0 \): indeed, \( f''(x) = \frac{e^x(e^x - 1)}{(e^x + 1)^2} > 0 \). Thus by Jensen’s inequality applied to \( f(x_1), \ldots, f(x_n) \), we get

\[
\frac{1}{r_1 + 1} + \cdots + \frac{1}{r_n + 1} \geq \frac{1}{\sqrt{r_1 \cdots r_n + 1}}.
\]

11. The given inequality is equivalent to \( x^3(x + 1) + y^3(y + 1) + z^3(z + 1) \geq \frac{4}{3}(x + 1)(y + 1)(z + 1) \). By the A-G mean inequality, it will be enough to prove a stronger inequality:

\[
x^4 + x^3 + y^4 + y^3 + z^4 + z^3 \geq \frac{1}{4}[(x + 1)^3 + (y + 1)^3 + (z + 1)^3].
\]

If we set \( S_k = x^k + y^k + z^k \), (1) takes the form \( S_4 + S_3 \geq \frac{1}{4}S_3^3 + \frac{3}{4}S_2^2 + \frac{3}{4}S_1 + \frac{3}{4} \).

Note that by the A-G mean inequality, \( S_1 = x + y + z \geq 3 \). Thus it suffices to prove the following:

If \( S_1 \geq 3 \) and \( m > n \) are positive integers, then \( S_m \geq S_n \).

This can be shown in many ways. For example, by Hölder’s inequality,

\[
(x^m + y^m + z^m)^{n/m} = (1 + 1 + 1)^{(m-n)/m} \geq x^n + y^n + z^n.
\]

(Another way is using the Chebyshev inequality: if \( x \geq y \geq z \) then \( x^{k-1} \geq y^{k-1} \geq z^{k-1} \); hence \( S_k = x \cdot x^{k-1} + y \cdot y^{k-1} + z \cdot z^{k-1} \geq \frac{1}{3}S_1 S_{k-1} \), and the claim follows by induction.)
Second solution. Assume that 

\[
\frac{1}{(x+1)(z+1)} \geq \frac{1}{(y+1)(z+1)}.
\]

Hence Chebyshev’s inequality gives that

\[
\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} 
\geq \frac{1}{3} (x^3 + y^3 + z^3) \cdot (3 + x + y + z)
\]

Now if we put \( x + y + z = 3S \), we have \( x^3 + y^3 + z^3 \geq 3S \) and \( (1+x)(1+y)(1+z) \leq (1+a)^3 \) by the A-G mean inequality. Thus the needed inequality reduces to

\[
\frac{6S^3}{(1+S)^3} \geq \frac{3}{4},
\]

which is obviously true because \( S \geq 1 \).

Remark. Both these solutions use only that \( x + y + z \geq 3 \).

12. The assertion is clear for \( n = 0 \). We shall prove the general case by induction on \( n \). Suppose that \( c(m,i) = c(m,m-i) \) for all \( i \) and \( m \leq n \). Then by the induction hypothesis and the recurrence formula we have \( c(n+1,k) = 2^k c(n,k) + c(n,k-1) \) and \( c(n+1,n+1-k) = 2^{n+1-k} c(n,n+1-k) + c(n,n-k) = 2^{n+1-k} c(n,k-1) + c(n,k) \). Thus it remains only to show that

\[
(2^k - 1) c(n,k) = (2^{n+1-k} - 1) c(n,k-1).
\]

We prove this also by induction on \( n \). By the induction hypothesis,

\[
c(n-1,k) = \frac{2^{n-k} - 1}{2^k - 1} c(n-1,k-1)
\]

and

\[
c(n-1,k-2) = \frac{2^{k-1} - 1}{2^{n+1-k} - 1} c(n-1,k-1).
\]

Using these formulas and the recurrence formula we obtain

\[
(2^k - 1) c(n,k) - (2^{n+1-k} - 1) c(n,k-1) = (2^k - 2^k) c(n-1,k) - (2^n - 3 \cdot 2^{k-1} + 1) c(n-1,k-1) - (2^{n+1-k} - 1) c(n-1,k-2) = (2^n - 2^k) c(n-1,k-1) - (2^n - 3 \cdot 2^{k-1} + 1) c(n-1,k-1) - (2^{k-1} - 1) c(n-1,k-1) = 0.
\]

This completes the proof.

Second solution. The given recurrence formula resembles that of binomial coefficients, so it is natural to search for an explicit formula of the form

\[
c(n,k) = \frac{F(n)}{F(k)F(n-k)},
\]

where \( F(m) = f(1)f(2) \cdots f(m) \) (with \( F(0) = 1 \)) and \( f \) is a certain function from the natural numbers to the real numbers. If there is such an \( f \), then \( c(n,k) = c(n,n-k) \) follows immediately.

After substitution of the above relation, the recurrence equivalently reduces to

\[
f(n+1) = 2^k f(n-k+1) + f(k).\]

It is easy to see that \( f(m) = 2^{m-1} \) satisfies this relation.

Remark. If we introduce the polynomial \( P_n(x) = \sum_{k=0}^n c(n,k)x^k \), the recurrence relation gives \( P_0(x) = 1 \) and \( P_{n+1}(x) = xP_n(x) + P_n(2x) \). As a consequence of the problem, all polynomials in this sequence are symmetric, i.e., \( P_n(x) = x^n P_n(x^{-1}) \).
13. Denote by $F$ the set of functions considered. Let $f \in F$, and let $f(1) = a$. Putting $n = 1$ and $m = 1$ we obtain $f(f(z)) = a^2z$ and $f(ax^2) = f(z)^2$ for all $z \in \mathbb{N}$. These equations, together with the original one, imply $f(x)^2f(y)^2 = f(x)^2f(ay^2) = f(x^2f(f(ay^2))) = f(x^2a^3y^2) = f(aaxy^2) = f(axy)^2$, or $f(axy) = f(x)f(y)$ for all $x, y \in \mathbb{N}$. Thus $f(ax) = af(x)$, and we conclude that

$$af(xy) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{N}. \quad (1)$$

We now prove that $f(x)$ is divisible by $a$ for each $x \in \mathbb{N}$. In fact, we inductively get that $f(x)^k = a^{k-1}f(x^k)$ is divisible by $a^{k-1}$ for every $k$. If $p^\alpha$ and $p^\beta$ are the exact powers of a prime $p$ that divide $f(x)$ and $a$ respectively, we deduce that $k\alpha \geq (k - 1)\beta$ for all $k$, so we must have $\alpha \geq \beta$ for any $p$. Therefore $a | f(x)$.

Now we consider the function on natural numbers $g(x) = f(x)/a$. The above relations imply

$$g(1) = 1, \quad g(xy) = g(x)g(y), \quad g(g(x)) = x \quad \text{for all } x, y \in \mathbb{N}. \quad (2)$$

Since $g \in F$ and $g(x) \leq f(x)$ for all $x$, we may restrict attention to the functions $g$ only.

Clearly $g$ is bijective. We observe that $g$ maps a prime to a prime. Assume to the contrary that $g(p) = uv$, $u, v > 1$. Then $g(uv) = p$, so either $g(u) = 1$ and $g(v) = 1$. Thus either $g(1) = u$ or $g(1) = v$, which is impossible.

We return to the problem of determining the least possible value of $g(1998)$. Since $g(1998) = g(2 \cdot 3^3 \cdot 37) = g(2) \cdot g(3^3) \cdot g(37)$, and $g(2), g(3), g(37)$ are distinct primes, $g(1998)$ is not smaller than $2^3 \cdot 3 \cdot 5 = 120$.

On the other hand, the value of 120 is attained for any function $g$ satisfying (2) and $g(2) = 3, g(3) = 2, g(5) = 37, g(37) = 5$. Hence the answer is 120.

14. If $x^2y + x + y$ is divisible by $xy^2 + y + 7$, then so is the number $y(x^2y + x + y) - x(xy^2 + y + 7) = y^2 - 7x$.

If $y^2 - 7x \geq 0$, then since $y^2 - 7x < xy^2 + y + 7$, it follows that $y^2 - 7x = 0$. Hence $(x, y) = (7t^2, 7t)$ for some $t \in \mathbb{N}$. It is easy to check that these pairs really are solutions.

If $y^2 - 7x < 0$, then $7x - y^2 > 0$ is divisible by $xy^2 + y + 7$. But then $xy^2 + y + 7 \leq 7x - y^2 < 7x$, from which we obtain $y \leq 2$. For $y = 1$, we are led to $x + 8 \mid 7x - 1$, and hence $x + 8 \mid 7(x + 8) - (7x - 1) = 57$. Thus the only possibilities are $x = 11$ and $x = 49$, and the obtained pairs $(11, 1), (49, 1)$ are indeed solutions. For $y = 2$, we have $4x + 9 \mid 7x - 4$, so that $7(4x + 9) - 4(7x - 4) = 79$ is divisible by $4x + 9$. We do not get any new solutions in this case.

Therefore all required pairs $(x, y)$ are $(7t^2, 7t)$ ($t \in \mathbb{N}$), $(11, 1)$, and $(49, 1)$.

15. The condition is obviously satisfied if $a = 0$ or $b = 0$ or $a = b$ or $a, b$ are both integers. We claim that these are the only solutions.
Suppose that $a, b$ belong to none of the above categories. The quotient $a/b = [a]/[b]$ is a nonzero rational number: let $a/b = p/q$, where $p$ and $q$ are coprime nonzero integers.

Suppose that $p \notin \{-1, 1\}$. Then $p$ divides $[an]$ for all $n$, so in particular $p$ divides $[a]$ and thus $a = kp + \varepsilon$ for some $k \in \mathbb{N}$ and $0 \leq \varepsilon < 1$. Note that $\varepsilon \neq 0$, since otherwise $b = kq$ would also be an integer. It follows that there exists an $n \in \mathbb{N}$ such that $1 \leq n\varepsilon < 2$. But then $[na] = [knp + n\varepsilon] = knp + 1$ is not divisible by $p$, a contradiction. Similarly, $q \notin \{-1, 1\}$ is not possible. Therefore we must have $p, q = \pm 1$, and since $a \neq b$, the only possibility is $b = -a$. However, this leads to $[-a] = -[a]$, which is not valid if $a$ is not an integer.

16. Let $S$ be a set of integers such that for no four distinct elements $a, b, c, d \in S$, it holds that $20 \mid a + b - c - d$. It is easily seen that there cannot exist distinct elements $a, b, c, d$ with $a \equiv b$ and $c \equiv d$ (mod 20). Consequently, if the elements of $S$ give $k$ different residues modulo 20, then $S$ itself has at most $k + 2$ elements.

Next, consider these $k$ elements of $S$ with different residues modulo 20. They give $\frac{k(k-1)}{2}$ different sums of two elements. For $k \geq 7$ there are at least 21 such sums, and two of them, say $a + b$ and $c + d$, are equal modulo 20; it is easy to see that $a, b, c, d$ are distinct. It follows that $k$ cannot exceed 6, and consequently $S$ has at most 8 elements.

An example of a set $S$ with 8 elements is $\{0, 20, 40, 1, 2, 4, 7, 12\}$. Hence the answer is $n = 9$.

17. Initially, we determine that the first few values for $a_n$ are $1, 3, 4, 7, 10, 12, 13, 16, 19, 21, 22, 25$. Since these are exactly the numbers of the forms $3k + 1$ and $9k + 3$, we conjecture that this is the general pattern. In fact, it is easy to see that the equation $x + y = 3z$ has no solution in the set $K = \{3k + 1, 9k + 3 \mid k \in \mathbb{N}\}$. We shall prove that the sequence $\{a_n\}$ is actually this set ordered increasingly.

Suppose $a_n > 25$ is the first member of the sequence not belonging to $K$. We have several cases:

(i) $a_n = 3r + 2$, $r \in \mathbb{N}$. By the assumption, one of $r + 1, r + 2, r + 3$ is of the form $3k + 1$ (and smaller than $a_n$), and therefore is a member $a_i$ of the sequence. Then $3a_i$ equals $a_n + 1$, $a_n + 4$, or $a_n + 7$, which is a contradiction because $1, 4, 7$ are in the sequence.

(ii) $a_n = 9r$, $r \in \mathbb{N}$. Then $a_n + a_2 = 3(3r + 1)$, although $3r + 1$ is in the sequence, a contradiction.

(iii) $a_n = 9r + 6$, $r \in \mathbb{N}$. Then one of the numbers $3r + 3, 3r + 6, 3r + 9$ is a member $a_j$ of the sequence, and thus $3a_j$ is equal to $a_n + 3, a_n + 12$, or $a_n + 21$, where $3, 12, 21$ are members of the sequence, again a contradiction.

Once we have revealed the structure of the sequence, it is easy to compute $a_{1998}$. We have $1998 = 4 \cdot 499 + 2$, which implies $a_{1998} = 9 \cdot 499 + a_2 = 4494$. 


18. We claim that, if $2^n - 1$ divides $m^2 + 9$ for some $m \in \mathbb{N}$, then $n$ must be a power of 2. Suppose otherwise that $n$ has an odd divisor $d > 1$. Then $2^d - 1 \mid 2^n - 1$ is also a divisor of $m^2 + 9 = m^2 + 3^2$. However, $2^d - 1$ has some prime divisor $p$ of the form $4k - 1$, and by a well-known fact, $p$ divides both $m$ and 3. Hence $p = 3$ divides $2^d - 1$, which is impossible, because for $d$ odd, $2^d \equiv 2 \pmod{3}$. Hence $n = 2^r$ for some $r \in \mathbb{N}$.

Now let $n = 2^r$. We prove the existence of $m$ by induction on $r$. The case $r = 1$ is trivial. Now for any $r > 1$ note that $2^{2^r} - 1 = (2^{2^{r-1}} - 1)(2^{2^{r-1}} + 1)$. The induction hypothesis claims that there exists an $m_1$ such that $2^{2^{r-1}} - 1 \mid m_1^2 + 9$. We also observe that $2^{2^{r-1}} + 1 \mid m_2^2 + 9$ for simple $m_2 = 3 \cdot 2^{2^{r-2}}$. By the Chinese remainder theorem, there is an $m \in \mathbb{N}$ that satisfies $m \equiv m_1 \pmod{2^{2^{r-1}} - 1}$ and $m \equiv m_2 \pmod{2^{2^{r-1}} + 1}$. It is easy to see that this $m^2 + 9$ will be divisible by both $2^{2^{r-1}} - 1$ and $2^{2^{r-1}} + 1$, i.e., that $2^{2^r} - 1 \mid m^2 + 9$. This completes the induction.

19. For $n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_i$ are distinct primes and $\alpha_i$ natural numbers, we have $\tau(n) = (\alpha_1 + 1) \cdots (\alpha_r + 1)$ and $\tau(n^2) = (2\alpha_1 + 1) \cdots (2\alpha_r + 1)$. Putting $k_i = \alpha_i + 1$, the problem reduces to determining all natural values of $m$ that can be represented as

$$m = \frac{2k_1 - 1}{k_1} \cdot \frac{2k_2 - 1}{k_2} \cdots \frac{2k_r - 1}{k_r}.$$ 

(1)

Since the numerator $\tau(n^2)$ is odd, $m$ must be odd too. We claim that every odd $m$ has a representation of the form (1). The proof will be done by induction.

This is clear for $m = 1$. Now for every $m = 2k - 1$ with $k$ odd the result follows easily, since $m = \frac{2k - 1}{k} \cdot k$, and $k$ can be written as (1). We cannot do the same if $k$ is even; however, in the case $m = 4k - 1$ with $k$ odd, we can write it as $m = \frac{16k - 3}{6k - 1} \cdot \frac{6k - 1}{k} \cdot k$, and this works.

In general, suppose that $m = 2^t k - 1$, with $k$ odd. Following the same pattern, we can write $m$ as

$$m = \frac{2^t (2^t - 1) k - (2^t - 1)}{2^t - 1} \cdot \frac{4(2^t - 1) k - 3}{2^t - 1} \cdot \frac{2(2^t - 1) k - 1}{2^t - 1} \cdot k.$$

The induction is finished. Hence $m$ can be represented as $\frac{\tau(n^2)}{\tau(n)}$ if and only if it is odd.

20. We first consider the special case $n = 3^r$. Then the simplest choice $\frac{10^n - 1}{9} = 11 \cdots 1$ ($n$ digits) works. This can be shown by induction: it is true for $r = 1$, while the inductive step follows from $10^{3^r} - 1 = (10^{3^{r-1}} - 1)(10^{2 \cdot 3^{r-1}} + 10^{3^{r-1}} + 1)$, because the second factor is divisible by 3.

In the general case, let $k \geq n/2$ be a positive integer and $a_1, \ldots, a_{n-k}$ be nonzero digits. We have
\[ A = (10^k - 1)a_1a_2 \ldots a_{n-k} \]
\[ = a_1a_2 \ldots a_{n-k-1}a'_{n-k} \underbrace{99 \ldots 99}_2b_1b_2 \ldots b_{n-k-1}b'_{n-k}, \]
where \( a'_{n-k} = a_{n-k} - 1, \) \( b_i = 9 - a_i, \) and \( b'_{n-k} = 9 - a'_{n-k}. \) The sum of digits of \( A \) equals \( 9k \) independently of the choice of digits \( a_1, \ldots, a_{n-k}. \) Thus we need only choose \( k \geq \frac{n}{2} \) and digits \( a_1, \ldots, a_{n-k-1} \not\in \{0, 9\} \) and \( a_{n-k} \in \{0, 1\} \) in order for the conditions to be fulfilled. Let us choose
\[ k = \begin{cases} 3^r, & \text{if } 3^r < n \leq 2 \cdot 3^r \text{ for some } r \in \mathbb{Z}, \\ 2 \cdot 3^r, & \text{if } 2 \cdot 3^r < n \leq 3^{r+1} \text{ for some } r \in \mathbb{Z}; \end{cases} \]
and \( a_1a_2 \ldots a_{n-k} = 22 \ldots 2. \) The number
\[ A = \underbrace{22 \ldots 2199 \ldots 99}_{n-k-1} \underbrace{t1 \ldots t8}_{2k-n} \underbrace{a}_{n-k-1} \]
thus obtained is divisible by \( 2 \cdot (10^k - 1), \) which is, as explained above, divisible by \( 18 \cdot 3^r. \) Finally, the sum of digits of \( A \) is either 9 \( \cdot 3^r \) or 18 \( \cdot 3^r; \) thus \( A \) has the desired properties.

21. Such a sequence is obviously strictly increasing. We note that it must be unique. Indeed, given \( a_0, a_1, \ldots, a_{n-1}, \) then \( a_n \) is the least positive integer not of the form \( a_i + 2a_j + 4a_k, \) \( i, j, k < n. \) We easily get that the first few \( a_n \)'s are \( 0, 1, 8, 9, 64, 65, 72, 73, \ldots. \) Let \( \{c_n\} \) be the increasing sequence of all positive integers that consist of zeros and ones in base 8, i.e., those of the form \( t_0 + 2^3t_1 + \cdots + 2^qt_q \) where \( t_i \in \{0, 1\}. \) We claim that \( a_n = c_n. \) To prove this, it is enough to show that each \( m \in \mathbb{N} \) can be uniquely written as \( c_i + 2c_j + 4c_k. \) If \( m = t_0 + 2t_1 + \cdots + 2^r t_r, \) then \( m = c_i + 2c_j + 4c_k \) is obviously possible if and only if \( c_i = t_0 + 2^3t_3 + 2^6t_6 + \cdots, \) \( c_j = t_1 + 2^3t_4 + \cdots, \) and \( c_k = t_2 + 2^3t_5 + \cdots. \) Hence for \( n = s_0 + 2s_1 + \cdots + 2^rs_r \) we have \( a_n = s_0 + 8s_1 + \cdots + 8^rs_r. \) In particular, 1998 = 2 + 2^2 + 2^3 + 2^6 + 2^7 + 2^8 + 2^9 + 2^{10}, so \( a_{1998} = 8 + 8^2 + 8^3 + 8^6 + 8^7 + 8^8 + 8^9 + 8^{10} = 1227096648. \)

Second solution. Define \( f(x) = x^{a_0} + x^{a_1} + \cdots. \) Then the assumed property of \( \{a_n\} \) gives
\[ f(x)f(x^2)f(x^4) = \sum_{i,j,k} x^{a_i+2a_j+4a_k} = \sum_n x^n = \frac{1}{1-x}. \]
We also get as a consequence \( f(x^2)f(x^4)f(x^8) = \frac{1}{1-x^2}, \) which gives \( f(x) = (1 + x)f(x^8). \) Continuing this, we obtain
\[ f(x) = (1 + x)(1 + x^8)(1 + x^{8^2}) \cdots. \]
Hence the \( a_n \)'s are integers that have only 0's and 1's in base 8.
22. We can obviously change each $x$ into $\lfloor x \rfloor$ or $\lceil x \rceil$ so that the column sums remain unchanged. However, this does not necessarily match the row sums as well, so let us consider the sum $S$ of the absolute values of the changes in the row sums. It is easily seen that $S$ is even, and we want it to be 0. A row may have a higher or lower sum than desired. Let us mark a cell by $-$ if its entry $x$ was changed to $\lfloor x \rfloor$, and by $+$ if it was changed to $\lceil x \rceil$ instead. We call a row $R_2$ accessible from a row $R_1$ if there is a column $C$ such that $C \cap R_1$ is marked $+$ and $C \cap R_2$ is marked $-$. Note that a column containing a $+$ must contain a $-$ as well, because column sums are unchanged. Hence from each row with a higher sum we can access another row.

Assume that the row sum in $R_1$ is higher. If $R_1, R_2, \ldots, R_k$ is a sequence of rows such that $R_{i+1}$ is accessible from $R_i$ via some column $C_i$ and such that the row sum in $R_k$ is lower, then by changing the signs in $C_i \cap R_i$ and $C_i \cap R_{i+1}$ ($i = 1, 2, \ldots, k - 1$) we decrease $S$ by 2, leaving column sums unchanged. We claim that such a sequence of rows always exists.

Let $R$ be the union of all rows that are accessible from $R_1$, directly or indirectly; let $\overline{R}$ be the union of the remaining rows. We show that for any column $C$, the sum in $R \cap C$ is not higher. If $R \cap C$ contains no $+$’s, then this is clear. If $R \cap C$ contains a $+$, since the rows of $R$ are not accessible, the set $\overline{R} \cap C$ contains no $-$’s. It follows that the sum in $\overline{R} \cap C$ is not lower, and since column sums are unchanged, we again come to the same conclusion. Thus the total sum in $\overline{R}$ is not higher. Therefore, there is a row in $\overline{R}$ with too low a sum, justifying our claim.

23. (a) If $n$ is even, then every odd integer is unattainable. Assume that $n \geq 9$ is odd. Let $a$ be obtained by addition from some $b$, and $b$ from $c$ by multiplication. Then $a$ is $2c + 2$, $2c + n$, $nc + 2$, or $nc + n$, and is in every case congruent to $2c + 2$ modulo $n - 2$. In particular, if $a \equiv -2 \pmod{n - 2}$, then also $b \equiv -4$ and $c \equiv -2 \pmod{n - 2}$.

Now consider any $a = kn(n - 2) - 2$, where $k$ is odd. If it is attainable, but not divisible by 2 or $n$, it must have been obtained by addition. Thus all predecessors of $a$ are congruent to either $-2$ or $-4$ (mod $n - 2$), and none of them equals 1, a contradiction.

(b) Call an attainable number addy if the last operation is addition, and multy if the last operation is multiplication. We prove the following claims by simultaneous induction on $k$:

1. $n = 6k$ is both addy and multy;
2. $n = 6k + 1$ is addy for $k \geq 2$;
3. $n = 6k + 2$ is addy for $k \geq 1$;
4. $n = 6k + 3$ is addy;
5. $n = 6k + 4$ is multy for $k \geq 1$;
6. $n = 6k + 5$ is addy.

The cases $k \leq 1$ are easily verified. For $k \geq 2$, suppose all six statements hold up to $k - 1$. 
Since $6k - 3$ is addy, $6k$ is multy.
Next, $6k - 2$ is multy, so both $6k = (6k - 2) + 2$ and $6k + 1 = (6k - 2) + 3$ are addy.
Since $6k$ is multy, both $6k + 2$ and $6k + 3$ are addy.
Number $6k + 4 = 2 \cdot (3k + 2)$ is multy, because $3k + 2$ is addy (being either $6l + 2$ or $6l + 5$).
Finally, we have $6k + 5 = 3 \cdot (2k + 1) + 2$. Since $2k + 1$ is $6l + 1$, $6l + 3$, or $6l + 5$, it is addy except for 7. Hence $6k + 5$ is addy except possibly for 23. But $23 = ((1 \cdot 2 + 2) \cdot 2 + 2) \cdot 2 + 3$ is also addy.
This completes the induction. Now 1 is given and $2 = 1 \cdot 2$, $4 = 1 + 3$.
It is easily checked that 7 is not attainable, and hence it is the only unattainable number.

24. Let $f(n)$ be the minimum number of moves needed to monotonize any permutation of $n$ distinct numbers. Let us be given a permutation $\pi$ of \{1, 2, ..., $n$\}, and let $k$ be the first element of $\pi$. In $f(n - 1)$ moves, we can transform $\pi$ to either $(k, 1, 2, ..., k - 1, k + 1, ..., n)$ or $(k, n, n - 1, ..., k + 1, k - 1, ..., 1)$. Now the former can be changed to $(k, k - 1, 1, 2, ..., 1, k + 1, ..., n)$, which is then monotonized in the next move. Similarly, the latter also can be monotonized in two moves. It follows that $f(n) \leq f(n - 1) + 2$.
Thus we shall be done if we show that $f(5) \leq 4$.
First we note that $f(3) = 1$. Consider a permutation of \{1, 2, 3, 4\}. If either 1 or 4 is the first or the last element, we need one move to monotonize the other three elements, and at most one more to monotonize the whole permutation. Of the remaining four permutations, $(2, 1, 4, 3)$ and $(3, 4, 1, 2)$ can also be monotonized in two moves. The permutations $(2, 4, 1, 3)$ and $(3, 1, 4, 2)$ require 3 moves, but by this we can choose whether to change them into $(1, 2, 3, 4)$ or $(4, 3, 2, 1)$.
We now consider a permutation of \{1, 2, 3, 4, 5\}. If either 1 or 5 is in the first or last position, we can monotonize the rest in 3 moves, but in such a way that the whole permutation can be monotonized in the next move. If this is not the case, then either 1 or 5 is in the second or fourth position. Then we simply switch it to the outside in one move and continue as in the former case. Hence $f(5) = 4$, as desired.

25. We use induction on $n$. For $n = 3$, we have a single two-element subset \{i, j\} that is split by $(i, k, j)$ (where $k$ is the third element of $U$). Assume that the result holds for some $n \geq 3$, and consider a family $\mathcal{F}$ of $n - 1$ proper subsets of $U = \{1, 2, ..., n + 1\}$, each with at least 2 elements.
To continue the induction, we need an element $a \in U$ that is contained in all $n$-element subsets of $\mathcal{F}$, but in at most one of the two-element subsets. We claim that such an $a$ exists. Let $\mathcal{F}$ contain $k$ $n$-element subsets and $m$ 2-element subsets ($k + m \leq n - 1$). The intersection of the $n$-element subsets contains exactly $n + 1 - k \geq m + 2$ elements. On the other hand, at most $m$ elements belong to more than one 2-element subset, which justifies our claim.
Now let $A$ be the 2-element subset that contains $a$, if it exists; otherwise, let $A$ be any subset from $\mathcal{F}$ containing $a$. Excluding $a$ from all the subsets from $\mathcal{F} \setminus \{A\}$, we get at most $n - 2$ subsets of $U \setminus \{a\}$ with at least 2 and at most $n - 1$ elements. By the inductive hypothesis, we can arrange $U \setminus \{a\}$ so that we split all the subsets of $\mathcal{F}$ except $A$. It remains to place $a$, and we shall make a desired arrangement if we put it anywhere away from $A$.

26. Put $n = 2r + 1$. Since each of the $\binom{n}{2}$ pairs of judges agrees on at most two candidates, the total number of agreements is at most $k \binom{n}{2}$. On the other hand, if the $i$th candidate is passed by $x_i$ judges and failed by $n - x_i$ judges, then the number of agreements on this candidate equals

$$\left(\frac{x_i}{2}\right) + \left(\frac{n - x_i}{2}\right) = \frac{x_i^2 + (n - x_i)^2 - n}{2} = \frac{r^2 + (n - r)^2 - n}{2} = \frac{(n - 1)^2}{4}.$$

Therefore the total number of agreements is at least $\frac{m(n - 1)^2}{4}$, which implies that

$$k \binom{n}{2} \geq \frac{m(n - 1)^2}{4}, \quad \text{hence} \quad \frac{k}{m} \geq \frac{n - 1}{2n - 2}.$$

Remark. The obtained inequality is sharp. Indeed, if $m = \binom{2r + 1}{r}$ and each candidate is passed by a different subset of $r$ judges, we get equality. A similar example shows that the result is not valid for even $n$. In that case the weaker estimate $k \binom{n}{2} \geq \frac{m(n - 1)^2}{4}$ holds.

27. Since this is essentially a graph problem, we call the points and segments vertices and edges of the graph. We first prove that the task is impossible if $k \leq 4$.

Cases $k \leq 2$ are trivial. If $k = 3$, then among the edges from a vertex $A$ there are two of the same color, say $AB$ and $AC$, so we don’t have all the three colors among the edges joining $A, B, C$.

Now let $k = 4$, and assume that there is a desired coloring. Consider the edges incident with a vertex $A$. At least three of them have the same color, say blue. Suppose that four of them, $AB, AC, AD, AE$, are blue. There is a blue edge, say $BC$, among the ones joining $B, C, D, E$. Then four of the edges joining $A, B, C, D$ are blue, and we cannot complete the coloring. So, exactly three edges from $A$ are blue: $AB, AC, AD$. Also, of the edges connecting any three of the 6 vertices other than $A, B, C, D$, one is blue (because the edges joining them with $A$ are not so). By a classical result, there is a blue triangle $EFG$ with vertices among these six. Now one of $EB, EC, ED$ must be blue as well, because none of $BC, BD, CD$ is. Let it be $EB$. Then four of the edges joining $B, E, F, G$ are blue, which is impossible.

For $k = 5$ the task is possible. Label the vertices $0, 1, \ldots, 9$. For each color, we divide the vertices into four groups and paint in this color every edge
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joining two from the same group, as shown below. Then among any 5 vertices, 2 must belong to the same group, and the edge connecting them has the considered color.

yellow: 01 12 20 36 69 93 57 48
red: 23 34 42 58 81 15 79 60
blue: 4 55 66 4 7 00 33 7 9 1 8 2
green: 6 77 88 6 9 22 55 9 1 3 0 4
orange: 8 99 00 8 1 44 77 1 3 5 2 6.

A desired coloring can be made for \( k \geq 6 \) as well. Paint the edge \( ij \) in the \((i + j)\)th color for \( i < j \leq 8 \), and in the \(2i\)th color if \( j = 9 \) (the addition being modulo 9). We can ignore the edges painted with the extra colors. Then the edges of one color appear as five disjoint segments, so that any complete \( k \)-graph for \( k \geq 5 \) contains one of them.

28. Let \( A \) be the number of markers with white side up, and \( B \) the number of pairs of markers whose squares share a side.

We claim that \( A + B \) does not change its parity as the game progresses. Suppose that in some move we remove a marker that has exactly \( k \) neighbors, among them \( r \) with white side up (\( 0 \leq r \leq k \leq 4 \)). Of course, this marker has its black side up. When it is removed, the \( r \) white markers get black side up, while the \( k - r \) black ones become white. Thus \( A \) changes by \( k - 2r \). As for \( B \), it decreases by \( k \). It follows that \( A \) decreases by \( 2r \) and preserves its parity, as claimed.

Initially, \( A = mn - 1 \) and \( B = m(n - 1) + n(m - 1) \); hence \( A + B \) equals \( 3mn - m - n - 1 \). If we succeed in removing all the markers, we end up with \( A + B = 0 \). Hence \( 3mn - m - n - 1 = (m - 1)(n - 1) + 2(mn - 1) \) must be even, or equivalently at least one of \( m \) and \( n \) is odd.

On the other hand, the game can be finished successfully if \( m \) or \( n \) is odd. Assume that \( m \) is odd. As shown in the picture, we can arrive at the position (1) in \( m \) moves; with \( \frac{m+1}{2} \) moves we reduce it to the position \((1\frac{1}{2})\), and with the next \( \frac{m-1}{2} \) moves to the position (2). We continue until we empty all the columns.

\[
\begin{align*}
A &= mn - 1 \\
B &= m(n - 1) + n(m - 1)
\end{align*}
\]

\[
A + B = 3mn - m - n - 1
\]

\[
\text{if } m \text{ or } n \text{ is odd}
\]

\[
\text{if } m \text{ is odd}
\]

\[
A = mn - 1 \\
B = m(n - 1) + n(m - 1)
\]

\[
A + B = 3mn - m - n - 1
\]

\[
\text{if } m \text{ or } n \text{ is odd}
\]

\[
\text{if } m \text{ is odd}
\]
4.40 Solutions to the Shortlisted Problems of IMO 1999

1. Obviously $(1, p)$ (where $p$ is an arbitrary prime) and $(2, 2)$ are solutions and the only solutions to the problem for $x < 3$ or $p < 3$.

Let us now assume $x, p \geq 3$. Since $p$ is odd, $(p-1)^x + 1$ is odd, and hence $x$ is odd. Let $q$ be the largest prime divisor of $x$, which also must be odd. We have $q \mid x \mid x^{p-1} | (p-1)^x + 1 \Rightarrow (p-1)^x \equiv -1 \pmod{q}$. Also from Fermat’s little theorem $(p-1)^{q-1} \equiv 1 \pmod{q}$. Since $q-1$ and $x$ are coprime, there exist integers $\alpha, \beta$ such that $x\alpha = (q-1)\beta + 1$. We also note that $\alpha$ must be odd. We now have $p - 1 \equiv (p-1)(q-1)\beta + 1 \equiv (p-1)^{x\alpha} \equiv -1 \pmod{q}$ and hence $q \mid p \Rightarrow q = p$. Since $x$ is odd, $p \mid x$, and $x \leq 2p$, it follows $x = p$ for all $x, p \geq 3$. Thus

$$p^{p-1} \mid (p-1)^x + 1 = p^2 \cdot \left(p^{p-2} - \left(\frac{p}{1}\right)p^{p-1} - \cdots - \left(\frac{p}{p-2}\right) + 1\right).$$

Since the expression in parenthesis is not divisible by $p$, it follows that $p^{p-1} | p^2$ and hence $p \leq 3$. One can easily verify that $(3, 3)$ is a valid solution.

We have shown that the only solutions are $(1, p)$, $(2, 2)$, and $(3, 3)$, where $p$ is an arbitrary prime.

2. We first prove that every rational number in the interval $(1, 2)$ can be represented in the form $\frac{a^3 + b^3}{a^3 + d^3}$. Taking $b, d$ such that $b \neq d$ and $a = b + d$, we get $a^2 - ab + b^2 = a^2 - ad + d^2$ and

$$\frac{a^3 + b^3}{a^3 + d^3} = \frac{(a + b)(a^2 - ab + b^2)}{(a + d)(a^2 - ad + d^2)} = \frac{a + b}{a + d}.$$

For a given rational number $1 < m/n < 2$ we can select $a = m + n$ and $b = 2m - n$ such that along with $d = a - b$ we have $\frac{a+b}{a+d} = \frac{m}{n}$. This completes the proof of the first statement.

For $m/n$ outside of the interval we can easily select a rational number $p/q$ such that $\sqrt[3]{\frac{m}{n}} < \frac{p}{q} < \sqrt[3]{\frac{2m}{n}}$. In other words $1 < \frac{p^3 m}{q^3 n} < 2$. We now proceed to obtain $a, b$ and $d$ for $\frac{p^3 m}{q^3 n}$ as before, and we finally have

$$\frac{p^3 m}{q^3 n} = \frac{a^3 + b^3}{a^3 + d^3} \Rightarrow \frac{m}{n} = \frac{(aq)^3 + (bq)^3}{(ap)^3 + (dp)^3}.$$

Thus we have shown that all positive rational numbers can be expressed in the form $\frac{a^3 + b^3}{a^3 + d^3}$.

3. We first prove the following lemma.

Lemma. For $d, c \in \mathbb{N}$ and $d^2 \mid c^2 + 1$ there exists $b \in \mathbb{N}$ such that $d^2(d^2 + 1) \mid b^2 + 1$.

Proof. It is enough to set $b = c + d^2c - d^3 = c + d^2(c - d)$. 
Using the lemma it suffices to find increasing sequences $d_n$ and $c_n$ such that $c_n - d_n$ is an increasing sequence and $d_n^2 | c_n^2 + 1$. We then obtain the desired sequences $a_n$ and $b_n$ from $a_n = d_n^2$ and $b_n = c_n + d_n^2(c_n - d_n)$. It is easy to check that $d_n = 2^{2n} + 1$ and $c_n = 2^{n-1}$ satisfy the required conditions.

Hence we have demonstrated the existence of increasing sequences $a_n$ and $b_n$ such that $a_n(a_n + 1) | b_n^2 + 1$.

**Remark.** There are many solutions to this problem. For example, it is sufficient to prove that the Pell-type equation $5a_n(a_n + 1) = b_n^2 + 1$ has an infinity of solutions in positive integers. Alternatively, one can show that $a_n(a_n + 1)$ can be represented as a sum of two coprime squares for infinitely many $a_n$, which implies the existence of $b_n$.

4. (a) The fundamental period of $p$ is the smallest integer $d(p)$ such that $p | 10^{d(p)} - 1$.

Let $s$ be an arbitrary prime and set $N_s = 10^2s + 10^s + 1$. In that case $N_s ≡ 3 \pmod{9}$. Let $p_s \neq 37$ be a prime dividing $N_s/3$. Clearly $p_s \neq 3$.

We claim that such a prime exists and that $3 | d(p_s)$. The prime $p_s$ exists, since otherwise $N_s$ could be written in the form $N_s = 3 \cdot 37^k \equiv 3 \pmod{4}$, while on the other hand for $s > 1$ we have $N_s \equiv 1 \pmod{4}$. Now we prove $3 | d(p_s)$. We have $p_s | N_s | 10^{3s} - 1$ and hence $d(p_s) | 3s$. We cannot have $d(p_s) | s$, for otherwise $p_s | 10^s - 1 \Rightarrow p_s | (10^{2s} + 10^s + 1, 10^s - 1) = 3$; and we cannot have $d(p_s) | 3$, for otherwise $p_s | 10^3 - 1 = 999 = 3^3 \cdot 37$, both of which contradict $p_s \neq 3, 37$. It follows that $d(p_s) = 3s$. Hence for every prime $s$ there exists a prime $p_s$ such that $d(p_s) = 3s$. It follows that the cardinality of $S$ is infinite.

(b) Let $r = r(s)$ be the fundamental period of $p \in S$. Then $p | 10^{3r} - 1$, $p | 10^r - 1 \Rightarrow p | 10^{2r} + 10^r + 1$. Let $x_j = \frac{10^j - 1}{p}$ and $y_j = \{x_j\} = 0.a_j a_{j+1}a_{j+2} \ldots$. Then $a_j < 10y_j$, and hence

$$f(k, p) = a_k + a_{k+r} + a_{k+2r} < 10(y_k + y_{k+r} + y_{k+2r}).$$

We note that $x_k + x_{k+s(p)} + x_{k+2s(p)} = \frac{10^{k-1}N_p}{p}$ is an integer, from which it follows that $y_k + y_{k+s(p)} + y_{k+2s(p)} \in \mathbb{N}$. Hence $y_k + y_{k+s(p)} + y_{k+2s(p)} \leq 2$. It follows that $f(k, p) < 20$. We note that $f(2, 7) = 4 + 8 + 7 = 19$. Hence 19 is the greatest possible value of $f(k, p)$.

5. Since one can arbitrarily add zeros at the end of $m$, which increases divisibility by 2 and 5 to an arbitrary exponent, it suffices to assume 2, 5 \n. If $(n, 10) = 1$, there exists an integer $w \geq 2$ such that $10^w \equiv 1 \pmod{n}$. We also note that $10^i \equiv 1 \pmod{n}$ and $10^{i+w} \equiv 10 \pmod{n}$ for all integers $i$ and $j$. Let us assume that $m$ is of the form $m = \sum_{i=1}^{u} 10^i + \sum_{j=1}^{v} 10^{j+w+1}$ for integers $u, v \geq 0$ (where if $u$ or $v$ is 0, the corresponding sum is 0). Obviously, the sum of the digits of $m$ is equal to $u + v$, and also $m \equiv u + 10v \pmod{n}$. Hence our problem reduces to finding integers $u, v \geq 0$ such that $u + v = k$ and $n | u + 10v = k + 9v$. Since $(n, 9) = 1$, it follows that there exists some $v_0$ such that $0 \leq v_0 < n \leq k$ and $9v_0 \equiv 0 \pmod{9}$.
$-k \pmod{n} \Rightarrow n \mid k + 9v_0$. Taking this $v_0$ and setting $u_0 = k - v_0$ we obtain the desired parameters for defining $m$.

6. Let $N$ be the smallest integer greater than $M$. We take the difference of the numbers in the progression to be of the form $10^m + 1$, $m \in \mathbb{N}$. Hence we can take $a_n = a_0 + n(10^m + 1) = b_1b_2\ldots b_9$ where $a_0$ is the initial term in the progression and $b_1b_2\ldots b_9$ is the decimal representation of $a_n$. Since $2m$ is the smallest integer $x$ such that $10^x \equiv 1 \pmod{10^m + 1}$, it follows that $10^k \equiv 10^l \pmod{10^m + 1} \iff k \equiv l \pmod{2m}$. Hence

$$a_0 \equiv a_n = b_1b_2\ldots b_9 \equiv \sum_{i=0}^{2m-1} c_i10^i \pmod{10^m + 1},$$

where $c_i = b_i + b_{2m+i} + b_{4m+i} + \cdots \geq 0$ for $i = 0, 1, \ldots, 2m - 1$ (these $c_i$ also depend on $n$). We note that $\sum_{i=0}^{2m-1} c_i10^i$ is invariant modulo $10^m + 1$ for all $n$ and that $\sum_{i=0}^{2m-1} c_i = \sum_{j=0}^{s} b_j$ for a given $n$. Hence we must choose $a_0$ and $m$ such that $a_0$ is not congruent to any number of the form $\sum_{i=0}^{2m-1} c_i10^i$, where $c_0 + c_1 + \cdots + c_{2m-1} \leq N$ ($c_0, c_1, \ldots, c_{2m-1} \geq 0$).

The number of ways to select the nonnegative integers $c_0, c_1, \ldots, c_{2m-1}$ such that $c_0 + c_1 + \cdots + c_{2m-1} \leq N$ is equal to the number of strictly increasing sequences $0 \leq c_0 < c_0 + c_1 + 1 < c_0 + c_1 + c_2 + 2 + \cdots < c_0 + c_1 + \cdots + c_{2m-1} + 2m - 1 \leq N + 2m - 1$, which is equal to the number of $2m$-element subsets of $\{0, 1, 2, \ldots, N + 2m - 1\}$, which is $\binom{N+2m}{2m}$. For sufficiently large $m$ we have $\binom{N+2m}{2m} < 10^m$, and hence in this case one can select $a_0$ such that $a_0$ is not congruent to $\sum_{i=0}^{2m-1} c_i10^i$ modulo $10^m + 1$ for any set of integers $c_0, c_1, \ldots, c_{2m-1}$ such that $c_0 + c_1 + \cdots + c_{2m-1} \leq N$. Thus we have found the desired arithmetic progression.

7. We use the following simple lemma.

Lemma. Suppose that $M$ is the interior point of a convex quadrilateral $ABCD$. Then it follows that $MA + MB < AD + DC + CB$.

Proof. We repeatedly make use of the triangle inequality. The line $AM$, in addition to $A$, intersects the quadrilateral in a second point $N$. In that case $AM + MB < AN + NB < AD + DC + CB$.

We now apply this lemma in the following way. Let $D$, $E$, and $F$ be median points of $BC, AC$, and $AB$. Any point $M$ in the interior of $\triangle ABC$ is contained in at least two of the three convex quadrilaterals $ABDE$, $BCEF$, and $CAF D$. Let us assume without loss of generality that $M$ is in the interior of $BCEF$ and $CAF D$. In that case we apply the lemma to obtain $AM + CM < AF + FD + DC$ and $BM + CM < CE + EF + FB$ to obtain

$$CM + AM + BM + CM < AF + FD + DC + CE + EF + FB$$

$$= AB + AC + BC$$

from which the required conclusion immediately follows.
8. Let $A$, $B$, $C$, and $D$ be inverses of four of the five points, with the fifth point being the pole of the inversion. A separator through the pole transforms into a line containing two of the remaining four points such that the remaining two points are on opposite sides of the line. A separator not containing the pole transforms into a circle through three of the points with the fourth point in its interior. Let $K$ be the convex hull of $A, B, C$, and $D$. We observe two cases:

(i) $K$ is a quadrilateral, for example $ABCD$. In that case the four separators are the two diagonals and two circles $ABC$ and $ADC$ if $\angle A + \angle C < 180^\circ$, or $BAD$ and $BCD$ otherwise. The remaining six viable circles and lines are clearly not separators.

(ii) $K$ is a triangle, for example $ABC$ with $D$ in its interior. In that case the separators are lines $DA$, $DB$, $DC$ and the circle $ABC$. No other lines and circles qualify.

We have thus shown that any set of five points satisfying the stated conditions will have exactly four separators.

9. Let $r_{PQ}$ denote a reflection about the planar bisector of $PQ$ with $P, Q \in S$. Let $G$ be the centroid of $S$. From $r_{PQ}(S) = S$ it follows that $r_{PQ}(G) = G$. Hence $G$ belongs to the perpendicular bisector of $PQ$ and thus $GP = GQ$. Consequently the whole of $S$ lies on a sphere $\Sigma$ centered at $G$. We note the following two cases:

(a) $S$ is a subset of a plane $\pi$. In this case $S$ is included in a circle $k$, $G$ being its center. Hence its $n$ points form a convex polygon $A_1A_2\ldots A_n$. When applying $r_{A_iA_{i+2}}$ for some $0 < i < n - 1$ the point $A_{i+1}$ transforms into some point of $S$ lying on the same side of $A_iA_{i+1}$, which has to be $A_{i+1}$ itself. It thus follows that $A_iA_{i+1} = A_{i+1}A_{i+2}$ for all $0 < i < n - 1$ and hence $A_1A_2\ldots A_n$ is a regular $n$-gon.

(b) The points in $S$ are not coplanar. It follows that $S$ is a polyhedron $P$ inscribed in a sphere $\Sigma$ centered at $G$. By applying the previous case to the faces of the polyhedron, it follows that all faces are regular $n$-gons.

Let us take an arbitrary vertex $V$ and let $VV_1$, $VV_2$ and $VV_3$ be three consecutive edges stemming from $V$ ($V, V_1, V_2,$ and $V_3$ defining two adjacent faces of $P$). We now look at $r_{V_1V_3}$. Since this transformation leaves the half-planes $[V_1V_3, V]$ and $[V_1V_3, V]$ invariant and since $V_2$ and $V$ are the only points of $P$ on the respective half-planes, it follows that $r_{V_1V_3}$ leaves $V$ and $V_2$ invariant. This transform also swaps $V_1$ and $V_3$. Hence, the face determined by $VV_1V_2$ is transformed by $r_{V_1V_3}$ into the face $VV_3V_2$, and thus the two faces sharing $VV_2$ are congruent. We conclude that all faces are congruent and similarly that vertices are endpoints of the same number of edges; hence $P$ is a regular polyhedron.

Finally, we have to rule out $S$ being vertices of a cube, a dodecahedron, or an icosahedron. In all of these cases if we select two diametrically
opposite points $P$ and $Q$, then $S \setminus \{P, Q\}$ is not symmetric with respect to the bisector of $PQ$, which prevents $r_{PQ}$ from being an invariant transformation of $S$.

It thus follows that the only viable finite completely symmetric sets are vertices of regular $n$-gons, the tetrahedron, and the octahedron. It is not explicitly asked for, but it is easy to verify that all of these are indeed completely symmetric.

**Remark.** On the IMO, a simpler version of this problem was adopted, adding the condition that $S$ belongs to a plane and thus eliminating the need for the second case altogether.

10. We use the following lemma.

**Lemma.** Let $ABC$ be a triangle and $X \in AB$ such that $\overrightarrow{AX} : \overrightarrow{XB} = m : n$.

Then $(m + n) \cot \angle CXB = n \cot A - m \cot B$ and $m \cot \angle ACX = (n + m) \cot C + n \cot A$.

**Proof.** Let $CD$ be the altitude from $C$ and $h$ its length. Then using oriented segments we have $AX = AD + DX = h \cot A - h \cot \angle CXB$ and $BX = BD + DX = h \cot B + h \cot \angle CXB$. The first formula in the lemma now follows from $n \cdot AX = m \cdot BX$. The second formula immediately follows from the first part applied to the triangle $ACX$ and the point $X' \in AC$ such that $XX' \parallel BC$.

Let us set $\cot A = x$, $\cot B = y$, and $\cot C = z$. Applying the second formula in the lemma to $\triangle ABC$ and the point $X$, we obtain $4 \cot \angle ACX = 9z + 5x$. Applying the first formula in the lemma to $\triangle CXZ$ and the point $Y$ and using $\angle XYZ = 45^\circ$ and $\cot \angle CXZ = -y$, we obtain $3 \cot \angle XYZ = \cot \angle ACX - 2 \cot \angle CXZ = 9z + 5x + 2y \Rightarrow 5x + 8y + 9z = 12$.

We now use the well-known relation for cotangents of a triangle $xy + yz + xz = 1$ to get $9 = 9(x + y)z + 9xy = (x + y)(12 - 5x - 8z) + 9xy = 9 \Rightarrow (4y + x - 3)^2 + 9(x - 1)^2 = 0 \Rightarrow x = 1, y = \frac{1}{2}, z = \frac{1}{3}$. It follows that $x$, $y$, and $z$ have fixed values, and hence all triangles $T$ in $\Sigma$ are similar, with their smallest angle $A$ having cotangent 1 and thus being equal to $\angle A = 45^\circ$.

11. Let $\Omega(I, r)$ be the incircle of $\triangle ABC$. Let $D$, $E$, and $F$ denote the points where $\Omega$ touches $BC$, $AC$, and $AB$, respectively. Let $P$, $Q$, and $R$ denote the midpoints of $EF$, $DF$, and $DE$ respectively. We prove that $\Omega_a$ passes through $Q$ and $R$. Since $\triangle IQD \sim \triangle IDB$ and $\triangle IRD \sim \triangle IDC$, we obtain $IQ \cdot IB = IR \cdot IC = r^2$. We conclude that $B, C, Q,$ and $R$ lie on a single circle $\Gamma_a$. Moreover, since the power of $I$ with respect to $\Gamma_a$ is $r^2$, it follows for a tangent $IX$ from $I$ to $\Gamma_a$ that $X$ lies on $\Omega$ and hence $\Omega$ is perpendicular to $\Gamma_a$. From the uniqueness of $\Omega_a$ it follows that $\Omega_a = \Gamma_a$. Thus $\Omega_a$ contains $Q$ and $R$. Similarly $\Omega_b$ contains $P$ and $R$ and $\Omega_c$ contains $P$ and $Q$. Hence, $A' = P$, $B' = Q$ and $C' = R$. Therefore the radius of the circumcircle of $\triangle A'B'C'$ is half the radius of $\Omega$.

12. We first introduce the following lemmas.
**Lemma 1.** Let $ABC$ be a triangle, $I$ its incenter and $I_a$ the center of the excircle touching $BC$. Let $A'$ be the center of the arc $BC$ of the circumcircle not containing $A$. Then $A'B = A'C = A'I = A'I_a$.

*Proof.* The result follows from a straightforward calculation of the relevant angles.

**Lemma 2.** Let two circles $k_1$ and $k_2$ meet each other at points $X$ and $Y$ and touch a circle $k$ internally in points $M$ and $N$, respectively. Let $A$ be one of the intersections of the line $XY$ with $k$. Let $AM$ and $AN$ intersect $k_1$ and $k_2$ respectively at $C$ and $E$. Then $CE$ is a common tangent of $k_1$ and $k_2$.

*Proof.* Since $AC \cdot AM = AX \cdot AY = AE \cdot AN$, the points $M, N, E, C$ lie on a circle. Let $MN$ meet $k_1$ again at $Z$. If $M'$ is any point on the common tangent at $M$, then $\angle MCZ = \angle M'MZ = \angle M'MN = \angle MAN$ (as oriented angles), implying that $CZ \parallel AN$. It follows that $\angle ACE = \angle ANM = \angle CZM$. Hence $CE$ is tangent to $k_1$ and analogously to $k_2$.

In the main problem, let us define $E$ and $F$ respectively as intersections of $NA$ and $NB$ with $\Omega_2$. Then applying Lemma 2 we get that $CE$ and $DF$ are the common tangents of $\Omega_1$ and $\Omega_2$.

If the circles have the same radii, the result trivially holds. Otherwise, let $G$ be the intersection of $CE$ and $DF$. Let $O_1$ and $O_2$ be the centers of $\Omega_1$ and $\Omega_2$. Since $O_1D = O_1C$ and $\angle O_1DG = \angle O_1CG = 90^\circ$, it follows that $O_1$ is the midpoint of the shorter arc of the circumcircle of $\triangle CDG$. The center $O_2$ is located on the bisector of $\angle CGD$, since $\Omega_2$ touches both $GC$ and $GD$.

However, it also sits on $O_1$, and using Lemma 1 we obtain that $O_2$ is either at the incenter or at the excenter of $\triangle CDG$ opposite $G$. Hence, $\Omega_2$ is either the incircle or the excircle of $CDG$ and thus in both cases touches $CD$.

**Second solution.** Let $O$ be the center of $\Gamma$, and $r, r_1, r_2$ the radii of $\Gamma, \Gamma_1, \Gamma_2$. It suffices to show that the distance $d(O_2, CD)$ is equal to $r_2$. The homothety with center $M$ and ratio $r/r_1$ takes $\Gamma_1, C, D$ into $\Gamma, A, B$, respectively; hence $CD \parallel AB$ and $d(C, AB) = \frac{r}{r_1}d(M, AB)$. Let $O_1O_2$ meet $XY$ at $R$. Then $d(O_2, CD) = O_2R + \frac{r-r_1}{r}d(M, AB)$, i.e.,

$$d(O_2, CD) = O_2R + \frac{r-r_1}{r}[O_1O_2 - O_2R + r_1 \cos \angle OO_1O_2],$$

(1)

since $O, O_1$, and $M$ are collinear. We have $O_1X = O_1O_2 = r_1$, $OO_1 = r - r_1$, $OO_2 = r - r_2$, and $O_2X = r_2$. Using the cosine law in the triangles $OO_1O_2$ and $XO_1O_2$, we obtain that $\cos \angle OO_1O_2 = \frac{2r_1^2 - 2rr_1 + 2rr_2 - r_2^2}{2r_1(r - r_1)}$ and $O_2R = r_2^2/2r_1$. Substituting these values in (1) we get $d(O_2, CD) = r_2$. 

13. Let us construct a convex quadrilateral PQRS and an interior point $T$ such that $\triangle PTQ \cong \triangle AMB$, $\triangle QTR \sim \triangle AMD$, and $\triangle PTS \sim \triangle CMD$. We then have $TS = \frac{MD \cdot PT}{MC} = MD$ and $TR = \frac{TR \cdot TP}{TM} = \frac{MD \cdot MB \cdot MC}{MA \cdot MA \cdot MD} = \frac{MB}{MC}$ (using $MA = MC$). We also have $\angle STR = \angle BMC$ and therefore $\triangle RTS \sim \triangle BMC$. Now the relations between angles become

\[ \angle TPS + \angle TQR = \angle PTQ \quad \text{and} \quad \angle TPQ + \angle TSR = \angle PTS, \]

implying that $PQ \parallel RS$ and $QR \parallel PS$. Hence $PQRS$ is a parallelogram and hence $AB = PQ = RS$ and $QR = PS$. It follows that $\frac{BC}{MC} = \frac{RS}{TS} = \frac{AB}{MD} \Rightarrow AB \cdot CM = BC \cdot MD$ and $\frac{AD \cdot BM}{AM} = \frac{AD \cdot QT}{AM} = QR = PS = \frac{CD \cdot TS}{MD} = CD \Rightarrow BM \cdot AD = MA \cdot CD$.

14. We first introduce the same lemma as in problem 12 and state it here without proof.

**Lemma.** Let $ABC$ be a triangle and $I$ the center of its incircle. Let $M$ be the center of the arc $BC$ of the circumcircle not containing $A$. Then $MB = MC = MI$.

Let the circle $XO_1O_2$ intersect the circle $\Omega$ again at point $T$. Let $M$ and $N$ be respectively the midpoints of arcs $BC$ and $AC$, and let $P$ be the intersection of $\Omega$ and the line through $C$ parallel to $MN$. Then the lemma gives $MP = NC = NI = NO_1$ and $NP = MC = MI = MO_2$. Since $O_1$ and $O_2$ lie on $XN$ and $XM$ respectively, we have $\angle NTM = \angle NXM = \angle O_1XO_2 = \angle O_1TO_2$ and hence $\angle NTO_1 = \angle MTO_2$. Moreover, $\angle TNO_1 = \angle TNX = \angle TMN$, from which it follows that $\triangle O_1NT \sim \triangle O_2MT$. Thus $\frac{NT}{MT} = \frac{NO_1}{O_2} = \frac{MT}{MT} = \frac{MT}{NM} \Rightarrow MP \cdot MT = NP \cdot NT \Rightarrow S_{MPT} = S_{NPT}$. It follows that $TP$ bisects the segment $MN$, and hence it passes through $I$. We conclude that $T$ belongs to the line $PI$ and does not depend on $X$.

**Remark.** An alternative approach is to apply an inversion at point $C$. Points $O_1$ and $O_2$ become excenters of $\triangle AXC$ and $\triangle BXC$, and $T$ becomes the projection of $I_c$ onto $AB$.

15. For all $x_i = 0$ any $C$ will do, so we may assume the contrary. Since the equation is symmetric and homogeneous, we may assume $\sum x_i = 1$. The equation now becomes

\[ F(x_1, x_2, \ldots, x_n) = \sum_{i < j} x_i x_j (x_i^2 + x_j^2) = \sum_i x_i^2 \sum_{j \neq i} x_j = \sum_i x_i^3 (1 - x_i) = \sum_i f(x_i) \leq C, \]

where we define $f(x) = x^3 - x^4$. We note that for $x, y \geq 0$ and $x + y \leq 2/3$,

\[ f(x + y) + f(0) - f(x) - f(y) = 3xy(x + y) \left( \frac{2}{3} - x - y \right) \geq 0. \quad (1) \]

We note that if at least three elements of $\{x_1, x_2, \ldots, x_n\}$ are nonzero the condition of (1) always holds for the two smallest ones. Hence, applying (1) repeatedly, we obtain $F(x_1, x_2, \ldots, x_n) \leq F(a, 1 - a, 0, \ldots, 0) = \frac{1}{7} (2a(1 - a))(1 - 2a(1 - a)) \leq \frac{1}{8} = F\left( \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 \right)$. Thus we have $C = \frac{1}{8}$ (for all
n), and equality holds only when two $x_i$ are equal and the remaining ones are 0.

Second solution. Let $M = x_1^2 + x_2^2 + \cdots + x_n^2$. Using $ab \leq (a + 2b)^2/8$ we have

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq M \sum_{i < j} x_i x_j$$

$$\leq \frac{1}{8} \left( M + 2 \sum_{i < j} x_i x_j \right)^2 = \frac{1}{8} \left( \sum_{i=1}^n x_i \right)^4.$$

Equality holds if and only if $M = 2 \sum_{i < j} x_i x_j$ for all $i < j$, which holds if and only if $n - 2$ of the $x_i$ are zero and the remaining two are equal.

Remark. Problems (SL90-26) and (SL91-27) are very similar.

16. Let $C(A)$ denote the characteristic of an arrangement $A$. We shall prove that $\max C(A) = \frac{n+1}{n}$.

Let us prove first $C(A) \leq \frac{n+1}{n}$ for all $A$. Among elements $\{n^2 - n, n^2 - n + 1, \ldots, n^2\}$, by the pigeonhole principle, in at least one row and at least one column there exist two elements, and hence one pair in the same row or column that is not $(n^2 - n, n^2)$. Hence

$$C(A) \leq \max \left\{ \frac{n^2}{n^2 - n + 1}, \frac{n^2 - 1}{n^2 - n} \right\} = \frac{n^2 - 1}{n^2 - n} = \frac{n + 1}{n}.$$

We now consider the following arrangement:

$$a_{ij} = \begin{cases} i + n(j - i - 1) & \text{if } i < j, \\ i + n(n - i + j - 1) & \text{if } i \geq j. \end{cases}$$

We claim that $C(a) = \frac{n+1}{n}$. Indeed, in this arrangement no two numbers in the same row or column differ by less than $n - 1$, and in addition, $n^2$ and $n^2 - n + 1$ are in different rows and columns, and hence

$$C(A) \geq \frac{n^2 - 1}{n^2 - n} = \frac{n + 1}{n}.$$

17. A game is determined by the ordering $t_1, \ldots, t_N$ of the $N = \binom{n}{2}$ transpositions $(i, j)$ of the set $\{1, 2, \ldots, n\}$. The game is nice if the permutation $P = t_N t_{N-1} \ldots t_1$ has no fixed point, and tiresome if $P$ is the identity (denoted by $I$). Recall that every permutation can be written as a composition of disjoint cycles.

We claim that there exists a nice game if and only if $n \neq 3$.

For $n = 2$, $P_2 = t_1 = (1, 2)$ is obviously nice. For $n = 3$ each game has the form $P = (b, c)(a, c)(a, b) = (a, c)$ for an appropriate notation of the players, which cannot be nice. Now for $n \geq 4$ we define
18. Define \( f(x, y) = x^2 - xy + y^2 \). Let us assume that three such sets \( A, B, \) and \( C \) do exist and that w.l.o.g. \( 1, b, \) and \( c (c > b) \) are respectively their smallest elements.

**Lemma 1.** Numbers \( x, y, \) and \( x + y \) cannot belong to three different sets.

**Proof.** The number \( f(x, x + y) = f(y, x + y) \) must belong to both the set containing \( y \) and the set containing \( x \), a contradiction.

**Lemma 2.** The subset \( C \) contains a multiple of \( b \). Moreover, if \( kb \) is the smallest such multiple, then \( (k - 1)b \in B \) and \( (k - 1)b + 1, kb + 1 \in A \).

**Proof.** Let \( r \) be the residue of \( c \) modulo \( b \). If \( r = 0 \), the first statement automatically holds. Let \( 0 < r < b \). In that case \( r \in A \), and \( c - r \) is then not in \( B \) according to Lemma 1. Hence \( c - r \in A \) and since \( b \mid c - r \), it follows that \( b \mid f(c - r, b) \in C \), thus proving the first statement. It follows immediately from Lemma 1 that \( (k - 1)b \in B \).

Now by Lemma 1, \( (k - 1)b + 1 = kb - (b - 1) \) must be in \( A \); similarly, \( kb + 1 = [(k - 1)b + 1] + b \in A \) as well.

Let us show by induction that \( (nk - 1)b + 1, nkb + 1 \in A \) for all integers \( n \). The inductive basis has been shown in Lemma 2. Assuming that \( ((n - 1)k - 1)b + 1 \in A \) and \((n - 1)kb + 1 \in A \), we get that \((nk - 1)b + 1 = ((n - 1)k - 1)b + (k - 1)b = [((n - 1)k - 1)b + 1] + kb \) belongs to \( A \) and...
\[ nkb + 1 = ((nk - 1)b + 1) + b = ((n - 1)kb + 1) + kb = nkb + 1 \in A. \] This finishes the inductive step. In particular, \( f(kb, kb+1) = (kb+1)kb+1 \in A. \) However, since \( kb \in C, kb+1 \in A, \) it follows that \( f(kb, kb+1) \in B, \) which is a contradiction.

19. Let \( A = \{ f(x) \mid x \in \mathbb{R} \} \) and \( f(0) = c. \) Plugging in \( x = y = 0 \) we get \( f(-c) = f(c) + c - 1, \) hence \( c \neq 0. \) If \( x \in A, \) then taking \( x = f(y) \) in the original functional equation we get \( f(x) = \frac{x^2}{2} - \frac{x}{2} \) for all \( x \in A. \)

We now show that \( A = A = \{ x_1 - x_2 \mid x_1, x_2 \in A \} = \mathbb{R}. \) Indeed, plugging in \( y = 0 \) into the original equation gives us \( f(x-c) = f(x) = cx + f(c) - 1, \) an expression that evidently spans all the real numbers. Thus, each \( x \) can be represented as \( x = x_1 - x_2, \) where \( x_1, x_2 \in A. \) Plugging \( x = x_1 \) and \( f(y) = x_2 \) into the original equation gives us

\[ f(x) = f(x_1 - x_2) = f(x_1) + x_1x_2 + f(x_2) - 1 = c - \frac{x_1^2 + x_2^2}{2} + x_1x_2 = c - \frac{x^2}{2}. \]

Hence we must have \( c = \frac{c+1}{2}, \) which gives us \( c = 1. \) Thus \( f(x) = 1 - \frac{x^2}{2} \) for all \( x \in \mathbb{R}. \) It is easily checked that this function satisfies the original functional equation.

20. We first introduce some useful notation. An arrangement around the circle will be denoted by \( x = \{ x_1, x_2, \ldots, x_n \}, \) where the elements are arranged clockwise and \( x_1 \) is fixed to be the smallest number. We will call an arrangement balanced if \( x_1 \leq x_n \leq x_2 \leq x_{n-1} \leq x_3 \leq x_{n-2} \leq \cdots \) (the string of inequalities continues until all the elements are accounted for).

We will denote the permutation of \( x = \{ x_1, x_2, \ldots, x_n \} \) in ascending order by \( x' = \{ x'_1, x'_2, \ldots, x'_n \}. \) We will let \( f_i(x) = \{ f_i(x_1), f_i(x_2), \ldots, f_i(x_{n-1}) \} \) denote the arrangement after one iteration of the algorithm where \( x_i \) was the deleted element.

**Lemma 1.** If an arrangement \( x \) is balanced, then \( f_1(x) \) is also balanced.

**Proof.** In one iteration we have \( \{ x_1, \ldots, x_n \} \rightarrow \{ x_n + x_2, x_2 + x_3, \ldots, x_{n-1} + x_n \}. \) Since \( x_n \leq x_2 \leq x_{n-1} \leq x_3 \leq x_{n-2} \leq \cdots, \) it follows that \( x_n + x_2 \leq x_n + x_{n-1} \leq x_2 \leq x_3 \leq x_{n-1} + x_{n-2} \leq \cdots, \) which means that \( f_1(x) \) is balanced.

We will first show by induction that \( S_{\text{max}} \) can be reached by using the balanced initial arrangement \( \{ a_1, a_3, a_5, \ldots, a_6, a_4, a_2 \} \) and repeatedly deleting the smallest member. For \( n = 3 \) we have \( S_3 = a_2 + a_3, \) in accordance with the formula. Assuming that the formula holds for a given \( n, \) we note that for an arrangement \( x = \{ a_1, a_3, a_5, \ldots, a_6, a_4, a_2 \} \) the arrangement \( f_1(x) \) is also balanced. We now apply the induction hypothesis and use that \( \binom{n-2}{i} + \binom{n-2}{i-1} = \binom{n-1}{i}; \)

\[ S(x) = S(f_1(x)) = \sum_{k=2}^{n-1} \left( \frac{n-2}{[k/2]} - 1 \right) (a_k + a_{k+2}) + \left( \frac{n-2}{[n/2]} - 1 \right) (a_n + a_{n+1}) = S_{\text{max}}. \]
We now prove that every other arrangement yields a smaller value. We shall write \( \{x_1, \ldots, x_n\} \leq \{y_1, \ldots, y_n\} \) whenever \( x'_n + x'_{n-1} + \cdots + x'_i \leq y'_n + y'_{n-1} + \cdots + y'_i \) holds for all \( 1 \leq i \leq n \).

**Lemma 2.** Let \( x \) be an arbitrary arrangement and \( y \) a balanced arrangement, both of \( n \) elements, such that \( x \leq y \). Then it follows that \( f_i(x) \leq f_1(y) \), for all \( i \).

**Proof.** For any \( 1 \leq j \leq n-1 \) there exists \( k_j \) such that \( f_i(x) \leq x_{k_j} + x_{k_j+1} \) (assuming \( k_j + 1 = 1 \) if \( k_j = n-1 \)). Then we have

\[
f_i(x)_{n-1} + \cdots + f_i(x)_{n-j} = (x_{k1} + x_{k1+1}) + \cdots + (x_{k_j} + x_{k_j+1}) \\
\leq 2x_n + \cdots + 2x_{n-i+1} + x'_{n-i} + x'_{n-i-1} \\
= f_1(y)_{n-1} + \cdots + f_1(y)_{n-j}
\]

for all \( j \), and hence \( f_i(x) \leq f_1(y) \).

An immediate consequence of Lemma 2 is \( f^{n-2}(x) \leq f_1^{n-2}(y) \), implying

\[
S = f^n(x)_1 + f^n(x)_2 \leq f_1^n(y)_1 + f_1^n(y)_2 = S_{\max}(y).
\]

Thus the proof is finished.

21. Let us call \( f(n, s) \) the number of paths from \((0,0)\) to \((n,n)\) that contain exactly \( s \) steps. Evidently, for all \( n \) we have \( f(n,1) = f(2,2) = 1 \), in accordance with the formula. Let us thus assume inductively for a given \( n > 2 \) that for all \( s \) we have \( f(n,s) = \frac{1}{s} \binom{n-1}{s-1} \binom{n}{s} \). We shall prove that the given formula holds also for all \( f(n+1,s) \), where \( s \geq 2 \).

We say that an \((n+1,s)\)- or \((n+1,s+1)\)-path is related to a given \((n,s)\)-path if it is obtained from the given path by inserting a step \( EN \) between two moves or at the beginning or the end of the path. We note that by inserting the step between two moves that form a step one obtains an \((n+1,s)\)-path; in all other cases one obtains an \((n+1,s+1)\)-path. For each \((n,s)\)-path there are exactly \( 2n+1-s \) related \((n+1,s+1)\)-paths, and for each \((n,s+1)\)-path there are \( s+1 \) related \((n+1,s+1)\)-paths. Also, each \((n+1,s+1)\)-path is related to exactly \( s+1 \) different \((n,s)\)- or \((n,s+1)\)-paths. Thus:

\[
(s+1)f(n+1,s+1) = (2n+1-s)f(n,s) + (s+1)f(n,s+1)
\]

\[
= \frac{2n+1-s}{s} (n-1) \binom{n}{s-1} \binom{n-1}{s} + \binom{n}{s} \binom{n-1}{s}
\]

\[
= \binom{n}{s} \binom{n+1}{s},
\]

i.e., \( f(n+1,s+1) = \frac{1}{s+1} \binom{n}{s} \binom{n+1}{s} \). This completes the proof.

22. (a) Color the first, third, and fifth row red, and the remaining squares white. There in total \( n \) pieces and \( 3n \) red squares. Since each piece can cover at most three red squares, it follows that each piece colors exactly three red squares. Then it follows that the two white squares it covers must be on the same row; otherwise, the piece has to cover
at least three. Hence, each white row can be partitioned into pairs of squares belonging to the same piece. Thus it follows that the number of white squares in a row, which is $n$, must be even.

(b) Let $a_k$ denote the number of different tilings of a $5 \times 2k$ rectangle. Let $b_k$ be the number of tilings that cannot be partitioned into two smaller tilings along a vertical line (without cutting any pieces). It is easy to see that $a_1 = b_1 = 2$, $b_2 = 2$, $a_2 = 6 = 2 \cdot 3$, $b_3 = 4$, and subsequently, by induction, $b_{3k} \geq 4$, $b_{3k+1} \geq 2$, and $b_{3k+2} \geq 2$. We also have $a_k = b_k + \sum_{i=1}^{k-1} b_i a_{k-i}$. For $k \geq 3$ we now have inductively
\[
a_k > 2 + \sum_{i=1}^{k-1} 2a_{k-i} \geq 2 \cdot 3^{k-1} + 2a_{k-1} \geq 2 \cdot 3^k.
\]

23. Let $r(m)$ denote the rest period before the $m$th catch, $t(m)$ the number of minutes before the $m$th catch, and $f(n)$ as the number of flies caught in $n$ minutes. We have $r(1) = 1$, $r(2m) = r(m)$, and $r(2m+1) = f(m) + 1$. We then have by induction that $r(m)$ is the number of ones in the binary representation of $m$. We also have $t(m) = \sum_{i=1}^{m} r(i)$ and $f(t(m)) = m$. From the recursive relations for $r$ we easily derive $t(2m+1) = 2t(m)+m+1$ and consequently $t(2m) = 2t(m)+m-r(m)$. We then have, by induction on $p$, $t(2^pm) = 2^pt(m) + p \cdot m \cdot 2^{p-1} - (2^p-1)r(m)$.

(a) We must find the smallest number $m$ such that $r(m+1) = 9$. The smallest number with nine binary digits is $1111111112 = 511$; hence the required $m$ is 510.

(b) We must calculate $t(98)$. Using the recursive formulas we have $t(98) = 2t(49) + 49 - r(49)$, $t(49) = 2t(24) + 25$, and $t(24) = 8t(3) + 36 - 7r(3)$. Since we have $t(3) = 4$, $r(3) = 2$ and $r(49) = r(1100112) = 3$, it follows $t(24) = 54 \Rightarrow t(49) = 133 \Rightarrow t(98) = 312$.

(c) We must find $m_c$ such that $t(m_c) \leq 1999 < t(m_c + 1)$. One can estimate where this occurs by using the formula $t(2^p(2^q - 1)) = (p + q)2^{p+q-1} - p \cdot 2^{p+1} - q \cdot 2^p + q$, provable from the recursive relations. It suffices to note that $t(462) = 1993$ and $t(463) = 2000$; hence $m_c = 462$.

24. Let $S = \{0, 1, \ldots, N^2 - 1\}$ be the group of residues (with respect to addition modulo $N^2$) and $A$ an $n$-element subset. We will use $|X|$ to denote the number of elements of a subset $X$ of $S$, and $\overline{X}$ to refer to the complement of $X$ in $S$. For $i \in S$ we also define $A_i = \{a+i \mid a \in A\}$. Our task is to select $0 \leq i_1 < \cdots < i_N \leq N^2 - 1$ such that $|\bigcup_{j=1}^{N} A_{i_j}| \geq \frac{1}{2}|S|$. Each $x \in S$ appears in exactly $N$ sets $A_i$. We have
\[
\sum_{i_1 < \cdots < i_N} \left| \bigcap_{j=1}^N A_{i_j} \right| = \sum_{i_1 < \cdots < i_N} |\{ x \in S \mid x \notin A_{i_1}, \ldots, A_{i_N} \}| \\
= \sum_{x \in S} |\{ i_1 < \cdots < i_N \mid x \notin A_{i_1}, \ldots, A_{i_N} \}| \\
= \sum_{x \in S} \binom{N^2 - N}{N} = \binom{N^2 - N}{N} |S|.
\]

Hence
\[
\sum_{i_1 < \cdots < i_N} \left| \bigcup_{j=1}^N A_{i_j} \right| = \sum_{i_1 < \cdots < i_N} \left( |S| - \left| \bigcap_{j=1}^N \overline{A}_{i_j} \right| \right) \\
= \left( \binom{N^2}{N} - \binom{N^2 - N}{N} \right) |S|.
\]

Thus, by the pigeonhole principle, one can choose \( i_1 < \cdots < i_N \) such that
\[
\left| \bigcup_{j=1}^N A_{i_j} \right| \geq \left( 1 - \frac{(N^2 - N)}{N^2} \right) |S|.
\]

Since \( \frac{(N^2)}{(N^2 - N)} \geq \left( \frac{N^2}{N^2 - N} \right)^N \), it follows that \( \left| \bigcup_{j=1}^N A_{i_j} \right| \geq \frac{1}{2} |S| \); hence the chosen \( i_1 < \cdots < i_N \) are indeed the elements of \( B \) that satisfy the conditions of the problem.

25. Let \( n = 2k \). Color the cells neighboring the edge of the board black. Then color the cells neighboring the black cells white. Then in alternation color the still uncolored cells neighboring the white or black cells on the boundary the opposite color and repeat until all cells are colored.

We call the cells colored the same color in each such iteration a “frame.”

In the color scheme described, each cell (white or black) neighbors exactly two black cells. The number of black cells is \( 2k(k + 1) \), and hence we need to mark at least \( k(k + 1) \) cells.

On the other hand, going along each black-colored frame, we can alternately mark two consecutive cells and then not mark two consecutive cells. Every cell on the black frame will have one marked neighbor. One can arrange these sequences on two consecutive black frames such that each cell in the white frame in between has exactly one neighbor. Hence, starting from a sequence on the largest frame we obtain a marking that contains exactly half of all the black cells, i.e., \( k(k + 1) \) and neighbors every cell.

It follows that the desired minimal number of markings is \( k(k + 1) \).

Remark. For \( n = 4k - 1 \) and \( n = 4k + 1 \) one can perform similar markings to obtain minimal numbers \( 4k^2 - 1 \) and \( (2k + 1)^2 \), respectively.
26. We denote colors by capital initial letters. Let us suppose that there exists a coloring \( f : \mathbb{Z} \rightarrow \{R, G, B, Y\} \) such that for any \( a \in \mathbb{Z} \) we have \( f\{a, a + x, a + y, a + x + y\} = \{R, G, B, Y\} \). We now define a coloring of an integer lattice \( g : \mathbb{Z} \times \mathbb{Z} \rightarrow \{R, G, B, Y\} \) by the rule \( g(i, j) = f(xi + yj) \). It follows that every unit square in \( g \) must have its vertices colored by four different colors.

If there is a row or column with period 2, then applying the condition to adjacent unit squares, we get (by induction) that all rows or columns, respectively, have period 2.

On the other hand, taking a row to be not of period 2, i.e., containing a sequence of three distinct colors, for example \( GRY \), we get that the next row must contain in these columns \( YBG \), and the following \( GRY \), and so on. It would follow that a column in this case must have period 2. A similar conclusion holds if we start with an aperiodic column. Hence either all rows or all columns must have period 2.

Let us assume w.l.o.g. that all rows have a period of 2. Assuming w.l.o.g. \( \{g(0,0), g(1,0)\} = \{G, B\} \), we get that the even rows are painted with \( \{G, B\} \) and odd with \( \{Y, R\} \). Since \( x \) is odd, it follows that \( g(y,0) \) and \( g(0,x) \) are of different color. However, since \( g(y,0) = f(xy) = g(0,x) \), this is a contradiction. Hence the statement of the problem holds.

27. Denote \( A = \{0, 1, 2\} \) and \( B = \{0, 1, 3\} \). Let \( f_T(x) = \sum_{a \in T} x^a \). Then define \( F_T(x) = f_T(x) f_T(x^2) \cdots f_T(x^{p-1}) \). We can write \( F_T(x) = \sum_{i=0}^{p(p-1)} a_i x^i \), where \( a_i \) is the number of ways to select an array \( \{x_1, \ldots, x_{p-1}\} \) where \( x_i \in T \) for all \( i \) and \( x_1 + 2x_2 + \cdots + (p-1)x_{p-1} = i \). Let \( w = \cos(2\pi/p) + i \sin(2\pi/p) \), a \( p \)th root of unity. Noting that

\[
1 + w^j + w^{2j} + \cdots + w^{(p-1)j} = \begin{cases} p, & p \mid j, \\ 0, & p \nmid j, \end{cases}
\]

it follows that \( F_T(1) + F_T(w) + \cdots + F_T(w^{p-1}) = pE(T) \).

Since \( |A| = |B| = 3 \), it follows that \( F_A(1) = F_B(1) = 3^{p-1} \). We also have for \( p \nmid i, j \) that \( F_T(w^i) = F_T(w^j) \). Finally, we have

\[
F_A(w) = \prod_{i=1}^{p-1} (1 + w^i + w^{2i}) = \prod_{i=1}^{p-1} \frac{1 - w^{3i}}{1 - w^i} = 1.
\]

Hence, combining these results, we obtain

\[
E(A) = \frac{3^{p-1} + p - 1}{p} \quad \text{and} \quad E(B) = \frac{3^{p-1} + (p - 1)F_B(w)}{p}.
\]

It remains to demonstrate that \( F_B(w) \geq 1 \) for all \( p \) and that equality holds only for \( p = 5 \). Since \( E(B) \) is an integer, it follows that \( F_B(w) \) is an integer and \( F_B(w) \equiv 1 \pmod p \). Since \( f_B(w^{p-1}) = f_B(w^1) \), it follows that \( F_B(w) = |f_B(w)|^2 |f_B(w^2)|^2 \cdots |f_B(w^{(p-1)/2})|^2 > 0 \). Hence \( F_B(w) \geq 1 \).
It remains to show that \( F_B(w) = 1 \) if and only if \( p = 5 \). We have the formula \((x-w)(x-w^2)\cdots(x-w^{p-1}) = x^{p-1} + x^{p-2} + \cdots + x + 1 = \frac{x^p-1}{x-1}\).

Let \( f_B(x) = x^3 + x + 1 = (x - \lambda)(x - \mu)(x - \nu) \), where \( \lambda, \mu, \) and \( \nu \) are the three zeros of the polynomial \( f_B(x) \). It follows that

\[
F_B(w) = \left(\frac{\lambda^p - 1}{\lambda - 1}\right) \left(\frac{\mu^p - 1}{\mu - 1}\right) \left(\frac{\nu^p - 1}{\nu - 1}\right) = -\frac{1}{3} (\lambda^p - 1)(\mu^p - 1)(\nu^p - 1),
\]

since \((\lambda - 1)(\mu - 1)(\nu - 1) = -f_B(1) = -3\). We also have \( \lambda + \mu + \nu = 0 \), \( \lambda \mu \nu = -1 \), \( \lambda^2 + \mu^2 + \nu^2 = (\lambda + \mu + \nu)^2 - 2(\lambda \mu + \lambda \nu + \mu \nu) = -2 \). By induction (using that \((\lambda^r + \mu^r + \nu^r) + (\lambda^{r-2} + \mu^{r-2} + \nu^{r-2}) + (\lambda^{r-3} + \mu^{r-3} + \nu^{r-3}) = 0\), it follows that \( \lambda^r + \mu^r + \nu^r \) is an integer for all \( r \in \mathbb{N} \).

Let us assume \( F_B(x) = 1 \). It follows that \((\lambda^p - 1)(\mu^p - 1)(\nu^p - 1) = -3\). Hence \( \lambda^p, \mu^p, \nu^p \) are roots of the polynomial \( p(x) = x^3 - qx^2 + (1+q)x + 1 \), where \( q = \lambda^p + \mu^p + \nu^p \). Since \( f_B(x) \) is an increasing function in real numbers, it follows that it has only one real root (w.l.o.g.) \( \lambda \), the other two roots being complex conjugates. From \( f_B(-1) < 0 < f_B(-1/2) \) it follows that \(-1 < \lambda < -1/2\). It also follows that \( \lambda^p \) is the \( x \) coordinate of the intersection of functions \( y = x^3 + x + 1 \) and \( y = q(x^2 - x) \). Since \( \lambda < \lambda^p < 0 \), it follows that \( q > 0 \); otherwise, \( q(x^2 - x) \) intersects \( x^3 + x + 1 \) at a value smaller than \( \lambda \). Additionally, as \( p \) increases, \( \lambda^p \) approaches \( 0 \), and hence \( q \) must increase.

For \( p = 5 \) we have \( 1+w+w^3 = -w^2(1+w^2) \) and hence \( G(w) = \prod_{i=1}^{p-1}(1 + w^{2i}) = 1 \). For a zero of \( f_B(x) \) we have \( x^5 = -x^3 - x^2 = -x^2 + x + 1 \) and hence \( q = \lambda^5 + \mu^5 + \nu^5 = -(\lambda^2 + \mu^2 + \nu^2) + (\lambda + \mu + \nu) + 3 = 5 \).

For \( p > 5 \) we also have \( q \geq 6 \). Assuming again \( F_B(x) = 1 \) and defining \( p(x) \) as before, we have \( p(-1) < 0, p(0) > 0, p(2) < 0, \) and \( p(x) > 0 \) for a sufficiently large \( x > 2 \). It follows that \( p(x) \) must have three distinct real roots. However, since \( \mu^p, \nu^p \in \mathbb{R} \Rightarrow \nu^p = \overline{\mu^p} = \mu^p \), it follows that \( p(x) \) has at most two real roots, which is a contradiction. Hence, it follows that \( F_B(x) > 1 \) for \( p > 5 \) and thus \( E(A) \leq E(B) \), where equality holds only for \( p = 5 \).
4.41 Solutions to the Shortlisted Problems of IMO 2000

1. In order for the trick to work, whenever $x + y = z + t$ and the cards $x, y$ are placed in different boxes, either $z, t$ are in these boxes as well or they are both in the remaining box.

Case 1. The cards $i, i + 1, i + 2$ are in different boxes for some $i$. Since $i + (i + 3) = (i + 1) + (i + 2)$, the cards $i$ and $i + 3$ must be in the same box; moreover, $i − 1$ must be in the same box as $i + 2$, etc. Hence the cards 1, 4, 7, . . . , 100 are placed in one box, the cards 2, 5, . . . , 98 are in the second, while 3, 6, . . . , 99 are in the third box. The number of different arrangements of the cards is 6 in this case.

Case 2. No three successive cards are all placed in different boxes. Suppose that 1 is in the blue box, and denote by $w$ and $r$ the smallest numbers on cards lying in the white and red boxes; assume w.l.o.g. that $w < r$. The card $w + 1$ is obviously not red, from which it follows that $r > w + 1$. Now suppose that $r < 100$. Since $w + r = (w − 1) + (r + 1)$, $r + 1$ must be in the blue box. But then $(r + 1) + w = r + (w + 1)$ implies that $w + 1$ must be red, which is a contradiction. Hence the red box contains only the card 100. Since $99 + w = 100 + (w − 1)$, we deduce that the card 99 is in the white box. Moreover, if any of the cards $k$, $2 \leq k \leq 99$, were in the blue box, then since $k + 99 = (k − 1) + 100$, the card $k − 1$ should be in the red box, which is impossible. Hence the blue box contains only the card 1, whereas the cards 2, 3, . . . , 99 are all in the white box.

In general, one box contains 1, another box only 100, while the remaining contains all the other cards. There are exactly 6 such arrangements, and the trick works in each of them.

Therefore the answer is 12.

2. Since the volume of each brick is 12, the side of any such cube must be divisible by 6.

Suppose that a cube of side $n = 6k$ can be built using $\frac{n^3}{12} = 18k^3$ bricks. Set a coordinate system in which the cube is given as $[0, n] \times [0, n] \times [0, n]$ and color in black each unit cube $[2p, 2p + 1] \times [2q, 2q + 1] \times [2r, 2r + 1]$. There are exactly $\frac{n^3}{27} = 27k^3$ black cubes. Each brick covers either one or three black cubes, which is in any case an odd number. It follows that the total number of black cubes must be even, which implies that $k$ is even. Hence $12 | n$.

On the other hand, two bricks can be fitted together to give a $2 \times 3 \times 4$ box. Using such boxes one can easily build a cube of side 12, and consequently any cube of side divisible by 12.

3. Clearly $m(S)$ is the number of pairs of point and triangle $(P_i, P_jP_kP_l)$ such that $P_i$ lies inside the circle $P_jP_kP_l$. Consider any four-element set $Sijkl = \{P_i, P_j, P_k, P_l\}$. If the convex hull of $Sijkl$ is the triangle $P_iP_jP_k$, then we have $a_i = a_j = a_k = 0, a_l = 1$. Suppose that the convex hull is
the quadrilateral $P_i P_j P_k P_l$. Since this quadrilateral is not cyclic, we may suppose that $\angle P_i + \angle P_k < 180^\circ < \angle P_j + \angle P_l$. In this case $a_i = a_k = 0$ and $a_j = a_l = 1$. Therefore $m(S_{ijkl})$ is 2 if $P_i, P_j, P_k, P_l$ are vertices of a convex quadrilateral, and 1 otherwise.

There are $\binom{\binom{n}{4}}{4}$ four-element subsets $S_{ijkl}$. If $a(S)$ is the number of such subsets whose points determine a convex quadrilateral, we have $m(S) = 2a(S) + \left(\binom{n}{4} - a(S)\right) = \binom{n}{4} + a(S) \leq 2\binom{n}{4}$. Equality holds if and only if every four distinct points of $S$ determine a convex quadrilateral, i.e. if and only if the points of $S$ determine a convex polygon. Hence $f(n) = 2\binom{n}{4}$ has the desired property.

4. By a good placement of pawns we mean the placement in which there is no block of $k$ adjacent unoccupied squares in a row or column. We can make a good placement as follows: Label the rows and columns with $0, 1, \ldots, n-1$ and place a pawn on a square $(i,j)$ if and only if $k$ divides $i + j + 1$. This is obviously a good placement in which the pawns are placed on three lines with $k, 2n - 2k$, and $2n - 3k$ squares, which adds up to $4n - 4k$ pawns in total. Now we shall prove that a good placement must contain at least $4n - 4k$ pawns. Suppose we have a good placement of $m$ pawns. Partition the board into nine rectangular regions as shown in the picture. Let $a, b, \ldots, h$ be the numbers of pawns in the rectangles $A, B, \ldots, H$ respectively. Note that each row that passes through $A, B,$ and $C$ either contains a pawn inside $B$, or contains a pawn in both $A$ and $C$. It follows that $a + c + 2b \geq 2(n - k)$. We similarly obtain that $c + e + 2d$, $e + g + 2f$, and $g + a + 2h$ are all at least $2(n - k)$. Adding and dividing by 2 yields $a + b + \cdots + h \geq 4(n - k)$, which proves the statement.

5. We say that a vertex of a nice region is convex if the angle of the region at that vertex equals $90^\circ$; otherwise (if the angle is $270^\circ$), we say that a vertex is concave. For a simple broken line $C$ contained in the boundary of a nice region $R$ we call the pair $(R, C)$ a boundary pair. Such a pair is called outer if the region $R$ is inside the broken line $C$, and inner otherwise. Let $B_i, B_o$ be the sets of inner and outer boundary pairs of nice regions respectively, and let $B = B_i \cup B_o$. For a boundary pair $b = (R, C)$ denote by $c_b$ and $v_b$ respectively the number of convex and concave vertices of $R$ that belong to $C$. We have the following facts:

(1) Each vertex of a rectangle corresponds to one concave angle of a nice region and vice versa. This correspondence is bijective, so $\sum_{b \in B} v_b = 4n$. 
(2) For a boundary pair \( b = (R, C) \) the sum of angles of \( R \) that are on \( C \) equals \( (c_b + v_b - 2)180^\circ \) if \( b \) is outer, and \( (c_b + v_b + 2)180^\circ \) if \( b \) is inner. On the other hand the sum of angles is obviously equal to \( c_b \cdot 90^\circ + v_b \cdot 270^\circ \). It immediately follows that \( c_b - v_b = \begin{cases} 4 & \text{if } b \in B_0, \\ -4 & \text{if } b \in B_1. \end{cases} \)

(3) Since every vertex of a rectangle appears in exactly two boundary pairs and each boundary pair contains at least one vertex of a rectangle, the number \( K \) of boundary pairs is less than or equal to \( 8n \).

(4) The set \( B_i \) is nonempty, because every boundary of the infinite region is inner.

Consequently, the sum of the numbers of the vertices of all nice regions is equal to

\[
\sum_{b \in B} (c_b + v_b) = \sum_{b \in B} (2v_b + (c_b - v_b)) \leq 2 \cdot 4n + 4(K - 1) - 4 \leq 40n - 8.
\]

6. Every integer \( z \) has a unique representation \( z = px + qy \), where \( x, y \in \mathbb{Z}, 0 \leq x \leq q - 1 \). Consider the region \( T \) in the \( xy \)-plane defined by the last inequality and \( px + qy \geq 0 \). There is a bijective correspondence between lattice points of this region and nonnegative integers given by \( (x, y) \mapsto z = px + qy \). Let us mark all lattice points of \( T \) whose corresponding integers belong to \( S \) and color in black the unit squares whose left-bottom vertices are at marked points. Due to the condition for \( S \), this coloring has the property that all points lying on the right or above a colored point are colored as well. In particular, since the point \((0, 0)\) is colored, all points above or on the line \( y = 0 \) are colored. What we need is the number of such colorings of \( T \).

The border of the colored subregion \( C \) of \( T \) determines a path from \((0, 0)\) to \((q, -p)\) consisting of consecutive unit moves either to the right or downwards. There are \( \binom{p+q}{p} \) such paths in total. We must find the number of such paths not going below the line \( l : px + qy = 0 \).

Consider any path \( \gamma = A_0A_1 \ldots A_{p+q} \) from \( A_0 = (0, 0) \) to \( A_{p+q} = (q, -p) \).

We shall see the path \( \gamma \) as a sequence \( G_1G_2 \ldots G_{p+q} \) of moves to the right \((R)\) or downwards \((D)\) with exactly \( p \) \( D \)'s and \( q \) \( R \)'s.

Two paths are said to be equivalent if one is obtained from the other by a circular shift of the corresponding sequence \( G_1G_2 \ldots G_{p+q} \). We note that all the \( p + q \) circular shifts of a path are distinct. Indeed, \( G_1 \ldots G_{p+q} \equiv G_{i+1} \ldots G_{i+p+q} \) would imply \( G_1 = G_{i+1} = G_{2i+1} = \cdots \) (where \( G_{j+p+q} = G_j \), so \( G_1 = \cdots = G_{p+q} \), which is impossible. Hence each equivalence class contains exactly \( p + q \) paths.

Let \( l_i, 0 \leq i < p + q \), be the line through \( A_i \) that is parallel to the line \( l \). Since \( \gcd(p, q) = 1 \), all these lines are distinct.

Let \( l_m \) be the unique lowest line among the \( l_i \)'s. Then the path \( G_{m+1}G_{m+2} \ldots G_{m+p+q} \) is above the line \( l \). Every other cyclic shift gives rise to a path having at least one vertex below the line \( l \). Thus each equiv-
alence class contains exactly one path above the line \( l \), so the number of such paths is equal to \( \frac{1}{p+q} (\frac{p+q}{p} \). Therefore the answer is \( \frac{1}{p+q} (\frac{p+q}{p} \).

7. Elementary computation gives \((a - 1 + \frac{1}{b}) (b - 1 + \frac{1}{c}) = ab - a + \frac{a}{c} - b + 1 - \frac{1}{c} + 1 - \frac{1}{b} + \frac{1}{bc} \). Using \( ab = \frac{1}{c} \) and \( \frac{1}{bc} = a \) we obtain

\[
\left( a - 1 + \frac{1}{b} \right) \left( b - 1 + \frac{1}{c} \right) = \frac{a}{c} - b - \frac{1}{b} + 2 \leq \frac{a}{c},
\]

since \( b + \frac{1}{b} \geq 2 \). Similarly we obtain

\[
\left( b - 1 + \frac{1}{c} \right) \left( c - 1 + \frac{1}{a} \right) \leq \frac{b}{a} \text{ and } \left( c - 1 + \frac{1}{a} \right) \left( a - 1 + \frac{1}{b} \right) \leq \frac{c}{b}.
\]

The desired inequality follows from the previous three inequalities. Equality holds if and only if \( a = b = c = 1 \).

8. We note that \( \{ta\} \) lies in \( \left( \frac{1}{3}, \frac{2}{3} \right) \) if and only if there is an integer \( k \) such that \( k + \frac{1}{3} < ta \leq k + \frac{2}{3} \), i.e., if and only if \( t \in I_k = \left( \frac{k+1/3}{a}, \frac{k+2/3}{a} \right) \) for some \( k \). Similarly, \( t \) should belong to the sets \( J_m = \left( \frac{m+1/3}{b}, \frac{m+2/3}{b} \right) \) and \( K_n = \left( \frac{n+1/3}{c}, \frac{n+2/3}{c} \right) \) for some \( m, n \). We have to show that \( I_k \cap J_m \cap K_n \) is nonempty for some integers \( k, m, n \).

The intervals \( K_n \) are separated by a distance \( \frac{2}{3c} \), and since \( \frac{2}{3c} < \frac{1}{3b} \), each of the intervals \( J_m \) intersects at least one of the \( K_n \)’s. Hence it is enough to prove that \( J_m \subset I_k \) for some \( k, m \).

Let \( u_m \) and \( v_m \) be the left and right endpoints of \( J_m \). Since \( av_m = au_m + \frac{a}{b} < au_m + \frac{1}{b} \), it will suffice to show that there is an integer \( m \) such that the fractional part of \( au_m \) lies in \( \left[ \frac{1}{3}, \frac{1}{2} \right] \).

Let \( a = da, b = d\beta, \gcd(\alpha, \beta) = 1 \). Setting \( m = d\mu \) we obtain that

\[
au_m = a \frac{m+1/3}{b} = \frac{am}{d\beta\bar{b}} + \frac{\alpha}{3\beta} \leq \frac{\alpha}{3\beta} \text{ for } \beta > 6 \text{ and } \text{ otherwise } \beta \leq 5 \text{, the only irreducible fractions } \frac{\alpha}{\beta} \text{ that satisfy } 2\alpha < \beta \text{ are } \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{2}{5} \text{, hence one can take } m \text{ to be } 1, 1, 2, 3 \text{ respectively. This justifies our claim.}
\]

9. Let us first solve the problem under the assumption that \( g(\alpha) = 0 \) for some \( \alpha \).

Setting \( y = \alpha \) in the given equation yields \( g(x) = (\alpha - 1) f(x) - x f(\alpha) \). Then the given equation becomes \( f(x + g(y)) = (\alpha + 1 - y) f(x) + (f(y) - f(\alpha)) x \), so setting \( y = \alpha + 1 \) we get \( f(x + n) = mx \), where \( n = g(\alpha + 1) \) and \( m = f(\alpha + 1) - f(\alpha) \). Hence \( f \) is a linear function, and consequently \( g \) is also linear. If we now substitute \( f(x) = ax + b \) and \( g(x) = cx + d \) in the given equation and compare the coefficients, we easily find that
\[ f(x) = \frac{cx - c^2}{1 + c} \quad \text{and} \quad g(x) = cx - c^2, \quad c \in \mathbb{R} \setminus \{-1\}. \]

Now we prove the existence of \( \alpha \) such that \( g(\alpha) = 0 \). If \( f(0) = 0 \) then putting \( y = 0 \) in the given equation we obtain \( f(x + g(0)) = g(x) \), so we can take \( \alpha = -g(0) \).

Now assume that \( f(0) = b \neq 0 \). By replacing \( x \) by \( g(x) \) in the given equation we obtain \( f(g(x) + g(y)) = g(x)f(y) - yg(g(x)) + g(g(x)) \) and, analogously, \( f(g(x) + g(y)) = g(y)f(x) - xf(g(y)) + g(g(y)) \). The given functional equation for \( x = 0 \) gives \( f(g(y)) = a - by \), where \( a = g(0) \). In particular, \( g \) is injective and \( f \) is surjective, so there exists \( c \in \mathbb{R} \) such that \( f(c) = 0 \). Now the above two relations yield

\[
g(x)f(y) - ay + g(g(x)) = g(y)f(x) - ax + g(g(y)). \quad (1)
\]

Plugging \( y = c \) in (1) we get \( g(g(x)) = g(c)f(x) - ax + g(c) + ac = kf(x) - ax + d \). Now (1) becomes \( g(x)f(y) + kf(x) = g(y)f(x) + kf(y) \).

For \( y = 0 \) we have \( g(x)b + kf(x) = af(x) + kb \), whence

\[
g(x) = \frac{a - k}{b} f(x) + k.
\]

Note that \( g(0) = a \neq k = g(c) \), since \( g \) is injective. From the surjectivity of \( f \) it follows that \( g \) is surjective as well, so it takes the value 0.

10. Clearly \( F(0) = 0 \) by (i). Moreover, it follows by induction from (i) that \( F(2^n) = f_{n+1} \) where \( f_n \) denotes the \( n \)th Fibonacci’s number. In general, if \( n = \epsilon_k 2^k + \epsilon_{k-1} 2^{k-1} + \cdots + \epsilon_1 2 + \epsilon_0 \) (where \( \epsilon_i \in \{0, 1\} \)), it is straightforward to verify that

\[
F(n) = \epsilon_k f_{k+1} + \epsilon_{k-1} f_k + \cdots + \epsilon_1 f_2 + \epsilon_0 f_1. \quad (1)
\]

We observe that if the binary representation of \( n \) contains no two adjacent ones, then \( F(3n) = F(4n) \). Indeed, if \( n = \epsilon_k 2^k + \cdots + \epsilon_0 2^0 \), where \( k_i + 1 - k_i \geq 2 \) for all \( i \), then \( 3n = \epsilon_k 2^{k+1} + 2^k \epsilon_{k-1} + \cdots + \epsilon_0 2^{k_0} + 2^{k_0} \).

According to this, in computing \( F(3n) \) each \( f_{i+1} \) in (1) is replaced by \( f_{i+1} + f_{i+2} = f_{i+3} \), leading to the value of \( F(4n) \).

We shall prove the converse: \( F(3n) \leq F(4n) \) holds for all \( n \geq 0 \), with equality if and only if the binary representation of \( n \) contains no two adjacent ones.

We prove by induction on \( m \geq 1 \) that this holds for all \( n \) satisfying \( 0 \leq n < 2^m \). The verification for the early values of \( m \) is direct. Assume it is true for a certain \( m \) and let \( 2^m \leq n \leq 2^{m+1} \). If \( n = 2^m + p, 0 \leq p < 2^m \), then (1) implies \( F(4n) = F(2^{m+2} + 4p) = f_{m+3} + F(4p) \). Now we distinguish three cases:

(i) If \( 3p < 2^m \), then the binary representation of \( 3p \) does not carry into that of \( 3 \cdot 2^m \). Then it follows from (1) and the induction hypothesis that
\[ F(3n) = F(3 \cdot 2^m) + F(3p) = f_{m+3} + F(3p) \leq f_{m+3} + F(4p) = F(4n). \]

Equality holds if and only if \( F(3p) = F(4p) \), i.e. \( p \) has no two adjacent binary ones.

(ii) If \( 2^m \leq 3p < 2^{m+1} \), then the binary representation of \( 3p \) carries 1 into that of \( 3 \cdot 2^m \). Thus \( F(3n) = f_{m+3} + (F(3p) - f_{m+1}) = f_{m+2} + F(3p) < f_{m+3} + F(4p) = F(4n). \)

(iii) If \( 2^{m+1} \leq p < 3 \cdot 2^m \), then the binary representation of \( 3p \) carries 10 into that of \( 3 \cdot 2^m \), which implies

\[ F(3n) = f_{m+3} + f_{m+1} + (F(3p) - f_{m+2}) = 2f_{m+1} + F(3p) < F(4n). \]

It remains to compute the number of integers in \([0, 2^m]\) with no two adjacent binary 1’s. Denote their number by \( u_m \). Among them there are \( u_{m-1} \) less than \( 2^{m-1} \) and \( u_{m-2} \) in the segment \([2^{m-1}, 2^m]\). Hence \( u_m = u_{m-1} + u_{m-2} \) for \( m \geq 3 \). Since \( u_1 = 2 = f_3 \), \( u_2 = 3 = f_4 \), we conclude that \( u_m = f_{m+2} = F(2^{m+1}) \).

11. We claim that for \( \lambda \geq \frac{1}{n-1} \) we can take all fleas as far to the right as we want. In every turn we choose the leftmost flea and let it jump over the rightmost one. Let \( d \) and \( \delta \) denote the maximal and the minimal distances between two fleas at some moment. Clearly, \( d \geq (n-1)\delta \). After the leftmost flea jumps over the rightmost one, the minimal distance does not decrease, because \( \lambda d \geq \delta \). However, the position of the leftmost flea moved to the right by at least \( \delta \), and consequently we can move the fleas arbitrarily far to the right after a finite number of moves.

Suppose now that \( \lambda < \frac{1}{n-1} \). Under this assumption we shall prove that there is a number \( M \) that cannot be reached by any flea. Let us assign to each flea the coordinate on the real axis in which it is settled. Denote by \( s_k \) the sum of all the numbers in the \( k \)th step, and by \( w_k \) the coordinate of the rightmost flea. Clearly, \( s_k \leq nw_k \). We claim that the sequence \( w_k \) is bounded.

In the \((k+1)\)th move let a flea \( A \) jump over \( B \), landing at \( C \), and let \( a, b, c \) be their respective coordinates. We have \( s_{k+1} - s_k = c - a \). Then by the given rule, \( \lambda(b - a) = c - b = s_{k+1} - s_k + a - b \), which implies \( s_{k+1} - s_k = (1 + \lambda)(b - a) = \frac{1 + \lambda}{\lambda}(c - b) \). Hence \( s_{k+1} - s_k \geq \frac{1 + \lambda}{\lambda}(w_{k+1} - w_k) \). Summing up these inequalities for \( k = 0, \ldots, n-1 \) yields \( s_n - s_0 \geq \frac{1 + \lambda}{\lambda}(w_n - w_0) \).

Now using \( s_n \leq nw_n \) we conclude that

\[ \left(1 + \frac{\lambda}{\lambda} - n\right)w_n \leq \frac{1 + \lambda}{\lambda}w_0 - s_0. \]

Since \( \frac{1 + \lambda}{\lambda} - n > 0 \), this proves the result.

12. Since \( D(A) = D(B) \), we can define \( f(i) > g(i) \geq 0 \) that satisfy \( b_i - b_{i-1} = a_{f(i)} - a_{g(i)} \) for all \( i \).

The number \( b_{i+1} - b_i \in D(B) = D(A) \) can be written in the form \( a_u - a_v \), \( u > v \geq 0 \). Then \( b_{i+1} - b_i = b_{i+1} - b_i + b_i - b_{i-1} \) implies
\[ a_{f(i+1)} + a_{f(i)} + a_v = a_{g(i+1)} + a_{g(i)} + a_u, \]
so the \( B_3 \) property of \( A \) implies that \( (f(i+1), f(i), v) \) and \( (g(i+1), g(i), u) \) coincide up to a permutation. It follows that either \( f(i+1) = g(i) \) or \( f(i) = g(i+1) \). Hence if we define \( R = \{ i \in \mathbb{N}_0 \mid f(i+1) = g(i) \} \) and \( S = \{ i \in \mathbb{N}_0 \mid f(i) = g(i+1) \} \) it holds that \( R \cup S = \mathbb{N}_0 \).

**Lemma.** If \( i \in R \), then also \( i + 1 \in R \).

**Proof.** Suppose to the contrary that \( i \in R \) and \( i + 1 \in S \), i.e., \( g(i) = f(i+1) = g(i+2) \). There are integers \( x \) and \( y \) such that \( b_{i+2} - b_{i-1} = a_x - a_y \). Then \( a_x - a_y = a_{f(i+2)} - a_{g(i+2)} + a_{f(i+1)} - a_{g(i+1)} + a_{f(i)} - a_{g(i)} = a_{f(i+2)} + a_{f(i)} - a_{g(i+1)} - a_{g(i),} \) so by the \( B_3 \) property \( (x, g(i+1), g(i)) \) and \( (y, f(i+2), f(i)) \) coincide up to a permutation. But this is impossible, since \( f(i+2), f(i) > g(i+2) = g(i) = f(i+1) > g(i+1) \).

This proves the lemma. Therefore if \( i \in R \neq \emptyset \), then it follows that every \( j > i \) belongs to \( R \). Consequently \( g(i) = f(i+1) > g(i+1) = f(i+2) > g(i+2) = f(i+3) > \cdots \) is an infinite decreasing sequence of nonnegative integers, which is impossible. Hence \( S = \mathbb{N}_0 \), i.e.,

\[ b_{i+1} - b_i = a_{f(i+1)} - a_{f(i)} \quad \text{for all } i \in \mathbb{N}_0. \]

Thus \( f(0) = g(1) < f(1) < f(2) < \cdots \), implying \( f(i) \geq i \). On the other hand, for any \( i \), there exist \( j, k \) such that \( 0 = a_{f(i)} - a_i = b_j - b_k = a_{f(j)} - a_{f(k)} \), so by the \( B_3 \) property \( i \in \{ f(i), f(k) \} \) is a value of \( f \). Hence we must have \( f(i) = i \) for all \( i \), which finally gives \( A = B \).

13. One can easily find \( n \)-independent polynomials for \( n = 0, 1 \). For example, \( P_0(x) = 2000x^{2000} + \cdots + 2x^2 + x + 0 \) is \( 0 \)-independent (for \( Q \in M(P_0) \) it suffices to exchange the coefficient 0 of \( Q \) with the last term), and \( P_1(x) = 2000x^{2000} + \cdots + 2x^2 + x - (1+2+\cdots+2000) \) is \( 1 \)-independent (since any \( Q \in M(P_1) \) vanishes at \( x = 1 \)). Let us show that no \( n \)-independent polynomials exist for \( n \notin \{ 0, 1 \} \).

Consider separately the case \( n = -1 \). For any set \( T \) we denote by \( S(T) \) the sum of elements of \( T \). Suppose that \( P(x) = x^{a_{2000}} + \cdots + a_1x + a_0 \) is \( -1 \)-independent. Since \( P(-1) = (a_0 + a_2 + \cdots + a_{2000}) - (a_1 + a_3 + \cdots + a_{1999}) \), this means that for any subset \( E \) of the set \( C = \{ a_0, a_1, \ldots, a_{2000} \} \) having 1000 or 1001 elements there exist elements \( e \in E \) and \( f \in C \setminus E \) such that \( S(E \cup \{ f \} \setminus \{ e \}) = S(E) - \frac{1}{2} S(C) = e - f \). We may assume w.l.o.g. that \( a_0 < a_1 < \cdots < a_{2000} \).

Suppose that \( E \) is a 1000-element subset of \( C \) containing \( b_0, b_1 \) but not \( b_{1999}, b_{2000} \). By the \( -1 \)-independence of \( P \) there exist \( e \in E \) and \( f \in C \setminus E \) such that \( S(E) - \frac{1}{2} S(C) = e - f \). The same must hold for the set \( E' = E \cup \{ b_{1999}, b_{2000} \} \setminus \{ b_0, b_1 \} \), so for some \( e' \in E' \) and \( f' \in C \setminus E' \) we have \( S(E') - \frac{1}{2} S(C) = e' - f' \). It follows that \( b_{1999} + b_{2000} - b_0 - b_1 = S(E') - S(E) = e + e' - f - f' \). Therefore the transposition \( e \leftrightarrow f \) must involve at least one of the elements \( b_0, b_1, b_{1999}, b_{2000} \).
There are 7994 possible transpositions involving one of these four elements. On the other hand, by (SL93-12) the subsets $E$ of $C$ containing $b_0, b_1$ but not $b_{1999}, b_{2000}$ give at least 998·999+1 distinct sums of elements, far exceeding 7994. This is a contradiction.

For the case $|n| \geq 2$ we need the following lemma.

**Lemma.** Let $n \geq 2$ be a natural number and $P(x) = a_m x^m + \cdots + a_1 x + a_0$ a polynomial with distinct coefficients. Then the set $\{Q(n) | Q \in M(P)\}$ contains at least $2^m$ elements.

**Proof.** We shall use induction on $m$. The statement is easily verified for $m = 1$. Assume w.l.o.g. that $a_m < \ldots < a_1 < a_0$. Consider two polynomials $Q_k$ and $Q_{k+1}$ of the form

$$Q_k(x) = a_m x^m + \cdots + a_k x^k + a_0 x^{k-1} + b_{k-1} x^{k-2} + \cdots + b_1,$$

$$Q_{k+1}(x) = a_m x^m + \cdots + a_{k+1} x^{k+1} + a_0 x^k + c_k x^{k-1} + \cdots + c_1,$$

where $(b_{k-1}, \ldots, b_1)$ and $(c_k, \ldots, c_1)$ are permutations of the sets $\{a_{k-1}, \ldots, a_1\}$ and $\{a_k, \ldots, a_1\}$ respectively. We claim that $Q_{k+1}(n) \geq Q_k(n)$. Indeed, since $a_0 - c_k \leq a_0 - a_k$ and $b_j - c_j < a_0 - a_k$ for $1 \leq j \leq n-1$, we have $Q_{k+1}(n) - Q_k(n) = (a_0 - a_k)n^k - (a_0 - c_k)n^{k-1} - (b_{k-1} - c_{k-1})n^{k-2} - \cdots - (b_1 - c_1) \geq (a_0 - a_k)(n^k - n^{k-1} - \cdots - n - 1) > 0$.

Furthermore, by the induction hypothesis the polynomials of the form $Q_k(x)$ take at least $2^{k-2}$ values at $x = n$. Hence the total number of values of $Q(n)$ for $Q \in M(P)$ is at least $1 + 1 + 2 + 2^2 + \cdots + 2^{m-1} = 2^m$.

Now we return to the main result. Suppose that $P(x) = a_{2000} x^{2000} + a_{1999} x^{1999} + a_0$ is an $n$-independent polynomial. Since $P_2(x) = a_{2000} x^{2000} + a_{1998} x^{1998} + \cdots + a_2 x^2 + a_0$ is a polynomial in $t = x^2$ of degree 1000, by the lemma it takes at least $2^{1000}$ distinct values at $x = n$. Hence $\{Q(n) | Q \in M(P)\}$ contains at least $2^{1000}$ elements. On the other hand, interchanging the coefficients $b_i$ and $b_j$ in a polynomial $Q(x) = b_{2000} x^{2000} + \cdots + b_0$ modifies the value of $Q$ at $x = n$ by $(b_i - b_j)(n^i - n^j) = (a_k - a_l)(n^i - n^j)$ for some $k, l$. Hence there are fewer than $2001^4$ possible modifications of the value at $n$. Since $2001^4 < 2^{1000}$, we have arrived at a contradiction.

14. The given condition is obviously equivalent to $a^2 \equiv 1 \pmod{n}$ for all integers $a$ coprime to $n$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of $n$ onto primes. Since by the Chinese remainder theorem the numbers coprime to $n$ can give any remainder modulo $p_i^{\alpha_i}$ except 0, our condition is equivalent to $a^2 \equiv 1 \pmod{p_i^{\alpha_i}}$ for all $i$ and integers $a$ coprime to $p_i$.

Now if $p_i \geq 3$, we have $2^2 \equiv 1 \pmod{p_i^{\alpha_i}}$, so $p_i = 3$ and $\alpha_i = 2$. If $p_j = 2$, then $2^3 \equiv 1 \pmod{2^{\alpha_j}}$ implies $\alpha_j \leq 3$. Hence $n$ is a divisor of $2^3 \cdot 3 = 24$.

Conversely, each $n \mid 24$ has the desired property.

15. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of $n$ onto primes ($p_1 < p_2 < \cdots < p_k$). Since $4n$ is a perfect cube, we deduce that $p_1 = 2$ and $\alpha_1 = 3 \beta_1 + 1$, $\alpha_2 = 3 \beta_2$, $\ldots$, $\alpha_k = 3 \beta_k$ for some integers $\beta_i \geq 0$. Using $d(n) = (\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_k + 1)$ we can rewrite the equation $d(n)^3 = 4n$ as
\[(3\beta_1 + 2) \cdot (3\beta_2 + 1) \cdots (3\beta_k + 1) = 2^{\beta_1 + 1} p_2^{\beta_2} \cdots p_k^{\beta_k}.\]

Since \(d(n)\) is not divisible by 3, it follows that \(p_i \geq 5\) for \(i \geq 2\). Thus the above equation is equivalent to

\[
\frac{3\beta_1 + 2}{2^{\beta_1 + 1}} = \frac{p_2^{\beta_2}}{3\beta_2 + 1} \cdots \frac{p_k^{\beta_k}}{3\beta_k + 1}. \tag{1}
\]

For \(i \geq 2\) we have \(p_i^{\beta_i} \geq (1 + 4)^\beta_i \geq 1 + 4\beta_i\); hence (1) implies that \(\frac{3\beta_1 + 2}{2^{\beta_1 + 1}} \geq 1\), which leads to \(\beta_1 \leq 2\).

For \(\beta_1 = 0\) or \(\beta_1 = 2\) we have that \(\frac{3\beta_1 + 2}{2^{\beta_1 + 1}} = 1\), and therefore \(\beta_2 = \cdots = \beta_k = 0\). This yields the solutions \(n = 2\) and \(n = 2^7 = 128\).

For \(\beta_1 = 1\) the left-hand side of (1) equals \(\frac{5}{2}\). On the other hand, if \(p_i > 5\) or \(\beta_i > 1\), then \(\frac{p_i^{\beta_i}}{3\beta_i + 1} > \frac{5}{4}\), which is impossible. We conclude that \(p_2 = 5\) and \(k = 2\), so \(n = 2000\).

Hence the solutions for \(n\) are 2, 128, and 2000.

16. More generally, we will prove by induction on \(k\) that for each \(k \in \mathbb{N}\) there exists \(n_k \in \mathbb{N}\) that has exactly \(k\) distinct prime divisors such that \(n_k \mid 2^{n_k} + 1\) and \(3 \mid n_k\).

For \(k = 1\), \(n_1 = 3\) satisfies the given conditions. Now assume that \(k \geq 1\) and \(n_k = 3^a m\) where \(3 \nmid m\), so that \(m\) has exactly \(k - 1\) prime divisors.

Then the number \(3n_k = 3^{a+1}m\) has exactly \(k\) prime divisors and \(2^{n_k} + 1 = (2^{n_k} - 2^{n_k} + 1)(2^{n_k} - 2^{n_k} + 1)\) is divisible by \(3n_k\), since \(3 \mid 2^{n_k} - 2^{n_k} + 1\). We shall find a prime \(p\) not dividing \(n_k\) such that \(n_{k+1} = 3pn_k\). It is enough to find \(p\) such that \(p \mid 2^{n_k} + 1\) and \(p \nmid 2^{n_k} + 1\).

Moreover, we shall show that for every integer \(a > 2\) there exists a prime number \(p\) that divides \(a^2 - a + 1\) but not \(a + 1\). To prove this we observe that \(\gcd(a^2 - a + 1, a + 1) = \gcd(3, a + 1)\). Now if \(3 \nmid a + 1\), we can simply take \(p = 3\); otherwise, if \(a = 3b - 1\), then \(a^2 - a + 1 = 9b^2 - 9b + 3\) is not divisible by \(3^2\); hence we can take for \(p\) any prime divisor of \(a^2 - a + 1/3\).

17. Trivially all triples \((a, 1, n)\) and \((1, m, n)\) are solutions. Assume now that \(a > 1\) and \(m > 1\).

If \(m\) is even, then \(a^m + 1 \equiv (-1)^m + 1 \equiv 2 \pmod{a + 1}\), which implies that \(a^m + 1 = 2^t\). In particular, \(a\) is odd. But this is impossible, since \(2 < a^m + 1 = (a^{m/2})^2 + 1 \equiv 2 \pmod{4}\). Hence \(m\) is odd.

Let \(p\) be an arbitrary prime divisor of \(m\) and \(m = pm_1\). Then \(a^m + 1 \mid (a + 1)^n \mid (a^{m_1} + 1)^n\), so \(b^p + 1 \mid (b + 1)^n\) for \(b = a^{m_1}\). It follows that

\[P = \frac{b^p + 1}{b + 1} = b^{p-1} - b^{p-2} + \cdots + 1 \mid (b + 1)^n.\]

Since \(P \equiv p \pmod{b + 1}\), we deduce that \(P\) has no prime divisors other than \(p\); hence \(P\) is a power of \(p\) and \(p \mid b + 1\). Let \(b = kp - 1, k \in \mathbb{N}\). Then by
the binomial formula we have \( b^i = (kp - 1)^i \equiv (-1)^{i+1}(kp - 1) \pmod{p^2} \), and therefore \( P \equiv -kp((p - 1) + (p - 2) + \cdots + 1) + p \equiv p \pmod{p^2} \). We conclude that \( P \leq p \). But we also have \( P \geq b^{p-1} - b^{p-2} \geq b^{p-2} > p \) for \( p > 3 \), so we must have \( P = p = 3 \). Since \( b = a^{m_1} \), we obtain \( a = 2 \) and \( m = 3 \). The triple \((2, 3, n)\) is indeed a solution if \( n \geq 2 \).

Hence the set of solutions is \( \{(a, 1, n) | a, m, n \in \mathbb{N}\} \cup \{(2, 3, n) | n \geq 2\} \).

**Remark.** This problem is very similar to (SL97-14).

18. It is known that the area of the triangle is \( S = pr = p^2/n \) and \( S = \sqrt{p(p - a)(p - b)(p - c)} \). It follows that \( p^3 = n^2(p - a)(p - b)(p - c) \), which by putting \( x = p - a, y = p - b, \) and \( z = p - c \) transforms into

\[
(x + y + z)^3 = n^2xyz. \tag{1}
\]

We will be done if we show that (1) has a solution in positive integers for infinitely many natural numbers \( n \). Let us assume that \( z = k(x + y) \) for an integer \( k > 0 \). Then (1) becomes \((k + 1)^3(x + y)^2 = kn^2xy\). Further, by setting \( n = 3(k + 1) \) this equation reduces to

\[
(k + 1)(x + y)^2 = 9kxy. \tag{2}
\]

Set \( t = x/y \). Then (2) has solutions in positive integers if and only if \((k + 1)(t + 1)^2 = 9kt\) has a rational solution, i.e., if and only if its discriminant \( D = k(5k - 4) \) is a perfect square. Setting \( k = u^2 \), we are led to show that \( 5u^2 - 4 = v^2 \) has infinitely many integer solutions. But this is a classic Pell-type equation, whose solution is every Fibonacci number \( u = F_{2i+1} \).

This completes the proof.

19. Suppose that a natural number \( N \) satisfies \( N = a_1^2 + \cdots + a_k^2 \), \( 2N = b_1^2 + \cdots + b_l^2 \), where \( a_i \), \( b_j \) are natural numbers such that none of the ratios \( a_i/a_j, b_i/b_j, a_i/b_j, b_j/a_i \) is a power of 2.

We claim that every natural number \( n > \sum_{i=0}^{4N-2}(2iN + 1)^2 \) can be represented as a sum of distinct squares. Suppose \( n = 4qN + r, 0 \leq r < 4N \). Then

\[
n = 4Ns + \sum_{i=0}^{r-1}(2iN + 1)^2
\]

for some positive integer \( s \), so it is enough to show that \( 4Ns \) is a sum of distinct even squares. Let \( s = \sum_{c=1}^{C}2^{2u_c} + \sum_{d=1}^{D}2^{2v_d+1} \) be the binary expansion of \( s \). Then

\[
4Ns = \sum_{c=1}^{C} \sum_{i=1}^{k}(2^{u_c+1}a_i)^2 + \sum_{d=1}^{D} \sum_{j=1}^{l}(2^{v_d+1}b_j)^2,
\]

where all the summands are distinct by the condition on \( a_i, b_j \).
It remains to choose an appropriate $N$: for example $N = 29$, because $29 = 5^2 + 2^2$ and $58 = 7^2 + 3^2$.

Second solution. It can be directly checked that every odd integer $67 < n \leq 211$ can be represented as a sum of distinct squares. For any $n > 211$ we can choose an integer $m$ such that $m^2 > \frac{n}{2}$ and $n - m^2$ is odd and greater than 67, and therefore by the induction hypothesis can be written as a sum of distinct squares. Hence $n$ is also a sum of distinct squares.

20. Denote by $k_1, k_2$ the given circles and by $k_3$ the circle through $A, B, C, D$. We shall consider the case that $k_3$ is inside $k_1$ and $k_2$, since the other case is analogous.

Let $AC$ and $AD$ meet $k_1$ at points $P$ and $R$, and $BC$ and $BD$ meet $k_2$ at $Q$ and $S$ respectively. We claim that $PQ$ and $RS$ are the common tangents to $k_1$ and $k_2$, and therefore $P, Q, R, S$ are the desired points.

The circles $k_1$ and $k_3$ are tangent to each other, so we have $DC || RP$.

Since

$$AC \cdot CP = XC \cdot CY = BC \cdot CQ,$$

the quadrilateral $ABQP$ is cyclic, implying that $\angle APQ = \angle ABQ = \angle ADC = \angle ARP$. It follows that $PQ$ is tangent to $k_1$. Similarly, $PQ$ is tangent to $k_2$.

21. Let $K$ be the intersection point of the lines $MN$ and $AB$.

Since $KA^2 = KM \cdot KN = KB^2$, it follows that $K$ is the midpoint of the segment $AB$, and consequently $M$ is the midpoint of $AB$. Thus it will be enough to show that $EM \perp PQ$, or equivalently that $EM \perp AB$. However, since $AB$ is tangent to the circle $G_1$ we have $\angle BAM = \angle ACM = \angle EAB$, and similarly $\angle ABM = \angle EBA$. This implies that the triangles $EAB$ and $MAB$ are congruent. Hence $E$ and $M$ are symmetric with respect to $AB$; hence $EM \perp AB$.

Remark. The proposer has suggested an alternative version of the problem: to prove that $EN$ bisects the angle $CND$. This can be proved by noting that $EANB$ is cyclic.

22. Let $L$ be the point symmetric to $H$ with respect to $BC$. It is well known that $L$ lies on the circumcircle $k$ of $\triangle ABC$. Let $D$ be the intersection point of $OL$ and $BC$. We similarly define $E$ and $F$. Then

$$OD + DH = OD + DL = OL = OE + EH = OF + FH.$$
We shall prove that $AD, BE$, and $CF$ are concurrent. Let line $AO$ meet $BC$ at $D'$. It is easy to see that $\angle O'D'O = \angle ODD'$; hence the perpendicular bisector of $BC$ bisects $DD'$ as well. Hence $BD = CD'$. If we define $E'$ and $F'$ analogously, we have $CE = AE'$ and $AF = BF'$. Since the lines $AD', BE', CF'$ meet at $O$, it follows that $\frac{BD'}{DC} \cdot \frac{CE'}{EA} \cdot \frac{AF'}{FB'} = 1$. This proves our claim by Ceva’s theorem.

23. First, suppose that there are numbers $(b_i, c_i)$ assigned to the vertices of the polygon such that

$$A_iA_j = b_i c_j - b_i c_j$$

for all $i, j$ with $1 \leq i \leq j \leq n$. (1)

In order to show that the polygon is cyclic, it is enough to prove that $A_1, A_2, A_3, A_i$ lie on a circle for each $i$, $4 \leq i \leq n$, or equivalently, by Ptolemy’s theorem, that $A_1A_2 \cdot A_3A_i + A_2A_3 \cdot A_iA_1 = A_1A_3 \cdot A_2A_i$. But this is straightforward with regard to (1).

Now suppose that $A_1A_2 \ldots A_n$ is a cyclic quadrilateral. By Ptolemy’s theorem we have $A_iA_j = A_2A_j \cdot \frac{A_1A_i}{A_1A_2} - A_2A_i \cdot \frac{A_1A_j}{A_1A_2}$ for all $i, j$. This suggests taking $b_i = -A_1A_2, b_i = A_2A_i$ for $i \geq 2$ and $c_i = \frac{A_1A_i}{A_1A_2}$ for all $i$. Indeed, using Ptolemy’s theorem, one easily verifies (1).

24. Since $\angle ABT = 180^\circ - \gamma$ and $\angle ACT = 180^\circ - \beta$, the law of sines gives $\frac{BP}{PC} = \frac{S_{ABT}}{S_{ACT}} = \frac{AB \sin \gamma}{AB \sin \beta} = \frac{c^2}{a^2}$, which implies $BP = \frac{c^2 a}{b^2 + c^2}$.

Denote by $M$ and $N$ the feet of perpendiculars from $P$ and $Q$ on $AB$. We have $\cot \angle ABQ = \frac{BN}{NQ} = \frac{BP + BM}{BP \sin \beta} = \frac{c + BP \sin \beta}{BP \sin \beta} = \frac{b^2 + c^2 + a^2 - b^2}{2 a c \sin \beta} = \frac{2 a^2 + c^2 + a^2 - b^2}{2 a c \sin \beta} = 2 \cot \alpha + 2 \cot \beta + \cot \gamma$. Similarly, $\cot \angle BAS = 2 \cot \alpha + 2 \cot \beta + \cot \gamma$; hence $\angle ABQ = \angle BAS$.

Now put $p = \cot \alpha$ and $q = \cot \beta$. Since $p + q \geq 0$, the A-G mean inequality gives us $\cot \angle ABQ = 2p + 2q + \frac{1-pq}{p+q} \geq 2p + 2q + \frac{1-(p+q)^2/4}{p+q} = \frac{7}{4}(p+q) + \frac{1}{p+q} \geq 2 \sqrt{\frac{7}{4}} = \sqrt{7}$. Hence $\angle ABQ \leq \arctan \frac{1}{\sqrt{7}}$. Equality holds if and only if $\cot \alpha = \cot \beta = \frac{1}{\sqrt{7}}$, i.e., when $a : b : c = 1 : 1 : \frac{1}{\sqrt{7}}$.

25. By the condition of the problem, $\triangle ADX$ and $\triangle BCX$ are similar. Then there exist points $Y'$ and $Z'$ on the perpendicular bisector of $AB$ such that $\triangle AY'Z'$ is similar and oriented the same as $\triangle ADX$, and $\triangle BY'Z'$ is (being congruent to $\triangle AY'Z'$) similar and oriented the same as $\triangle BCX$. Since then $AD/AY' = AX/AZ'$ and $\angle DAY' = \angle XAZ'$, $\triangle ADY'$ and $\triangle AXZ'$ are also similar, implying $\frac{AD}{AX} = \frac{DY'}{XZ'}$. Analogously, $\frac{BC}{BX} = \frac{CY'}{XZ'}$.

It follows from $\frac{AD}{AX} = \frac{BC}{BX}$ that $CY' = DY'$, which means that $Y'$ lies on the perpendicular bisector of $CD$. Hence $Y' \equiv Y$. 
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Now $\angle AYB = 2\angle AYZ' = 2\angle ADX$, as desired.

26. The problem can be reformulated in the following way: Given a set $S$ of ten points in the plane such that the distances between them are all distinct, for each point $P \in S$ we mark the point $Q \in S \setminus \{P\}$ nearest to $P$. Find the least possible number of marked points.

Observe that each point $A \in S$ is the nearest to at most five other points. Indeed, for any six points $P_1, \ldots, P_6$ one of the angles $P_iP_j$ is at most $60^\circ$, in which case $P_iP_j$ is smaller than one of the distances $AP_i, AP_j$. It follows that at least two points are marked.

Now suppose that exactly two points, say $A$ and $B$, are marked. Then $AB$ is the minimal distance of the points from $S$, so by the previous observation the rest of the set $S$ splits into two subsets of four points according to whether the nearest point is $A$ or $B$. Let these subsets be $\{A_1, A_2, A_3, A_4\}$ and $\{B_1, B_2, B_3, B_4\}$ respectively. Assume that the points are labelled so that the angles $A_1AA_4+1$ and $B_1BB_4+1$ are successively adjacent as well as the angles $A_1AB+1$ and $B_1BA$. Adding these relations yields $\angle A_1AB > \angle A_1B_1A$. Adding these two inequalities, we get

$$180^\circ > \angle A_1AB + \angle B_1BA > \angle A_1B_1B + \angle B_1A_1A;$$

hence the sum of the angles of the quadrilateral $ABB_1A_1$ is less than $360^\circ$, which is a contradiction. Thus at least 3 points are marked.

An example of a configuration in which exactly 3 gangsters are killed is shown below.

27. Denote by $\alpha_1, \alpha_2, \alpha_3$ the angles of $\triangle A_1A_2A_3$ at vertices $A_1, A_2, A_3$ respectively. Let $T_1, T_2, T_3$ be the points symmetric to $L_1, L_2, L_3$ with respect to $A_1I, A_2I$, and $A_3I$ respectively. We claim that $T_1T_2T_3$ is the desired triangle.
Denote by $S_1$ and $R_1$ the points symmetric to $K_1$ and $K_3$ with respect to $L_1L_3$. It is enough to show that $T_1$ and $T_3$ lie on the line $R_1S_1$. To prove this, we shall prove that $\angle K_1S_1T_1 = \angle K'S_1K_1$ for a point $K'$ on the line $K_1K_3$ such that $K_3$ and $K'$ lie on different sides of $K_1$. We show first that $S_1 \in A_1I$. Let $X$ be the point of intersection of lines $A_1I$ and $L_1L_3$. We see from the triangle $A_1L_3X$ that $\angle L_1XI = \alpha_3/2 = \angle L_1A_3I$, which implies that $L_1XA_3I$ is cyclic. We now have $\angle A_1XA_3 = 90^\circ = \angle A_1K_1A_3$; hence $A_1K_1A_3$ is also cyclic. It follows that $\angle K_1XI = \angle K_1A_3A_1 = \alpha_3 = 2\angle L_1XI$; hence $X_1L_1$ bisects the angle $K_1X_1I$. Hence $S_1 \in XI$ as claimed. Now we have $\angle K_1S_1T_1 = \angle K_1S_1L_1 + 2\angle L_1S_1X = \angle S_1K_1L_1 + 2\angle L_1K_1X$. It remains to prove that $K_1X$ bisects $\angle A_3K_1K'$. From the cyclic quadrilateral $A_1K_1X_3$ we see that $\angle XK_1A_3 = \alpha_1/2$. Since $A_1K_3K_1A_3$ is cyclic, we also have $\angle K'K_1A_3 = \alpha_1 = 2\angle XK_1A_3$, which proves the claim.
4.42 Solutions to the Shortlisted Problems of IMO 2001

1. First, let us show that such a function is at most unique. Suppose that \( f_1 \) and \( f_2 \) are two such functions, and consider \( g = f_1 - f_2 \). Then \( g \) is zero on the boundary and satisfies

\[
g(p, q, r) = \frac{1}{6} [g(p + 1, q - 1, r) + \cdots + g(p, q - 1, r + 1)],
\]

i.e., \( g(p, q, r) \) is equal to the average of the values of \( g \) at six points \( (p + 1, q - 1, r), \ldots \) that lie in the plane \( \pi \) given by \( x + y + z = p + q + r \). Suppose that \( (p, q, r) \) is the point at which \( g \) attains its maximum in absolute value on \( \pi \cap T \). The averaging property of \( g \) implies that the values of \( g \) at \( (p + 1, q - 1, r) \) etc. are all equal to \( g(p, q, r) \). Repeating this argument we obtain that \( g \) is constant on the whole of \( \pi \cap T \), and hence it equals 0 everywhere. Therefore \( f_1 \equiv f_2 \).

It remains to guess \( f \). It is natural to try \( f(p, q, r) = pqr \) first: it satisfies \( f(p, q, r) = \frac{1}{6} [f(p + 1, q - 1, r) + \cdots + f(p, q - 1, r + 1)] + \frac{p+q+r}{3} \). Thus we simply take

\[
f(p, q, r) = \frac{3}{p + q + r} f(p, q, r) = \frac{3pqr}{p + q + r}
\]

and directly check that it satisfies the required property. Hence this is the unique solution.

2. It follows from Bernoulli’s inequality that for each \( n \in \mathbb{N} \), \( (1 + \frac{1}{n})^n \geq 2 \), or \( \sqrt{n} \leq 1 + \frac{1}{n} \). Consequently, it will be enough to show that \( 1 + a_n > (1 + \frac{1}{n}) a_{n-1} \). Assume the opposite. Then there exists \( N \) such that for each \( n \geq N \),

\[
1 + a_n \leq \left(1 + \frac{1}{n}\right) a_{n-1}, \quad \text{i.e.,} \quad \frac{1}{n+1} + \frac{a_n}{n+1} \leq \frac{a_{n-1}}{n}.
\]

Summing for \( n = N, \ldots, m \) yields \( \frac{a_m}{m+1} \leq \frac{a_N}{N} - \left(\frac{1}{N+1} + \cdots + \frac{1}{m+1}\right) \). However, it is well known that the sum \( \frac{1}{N+1} + \cdots + \frac{1}{m+1} \) can be arbitrarily large for \( m \) large enough, so that \( \frac{a_m}{m+1} \) is eventually negative. This contradiction yields the result.

Second solution. Suppose that \( 1 + a_n \leq \sqrt{2} a_{n-1} \) for all \( n \geq N \). Set \( b_n = 2^{-(1+1/2+\cdots+1/n)} \) and multiply both sides of the above inequality to obtain \( b_n + b_n a_n \leq b_{n-1} a_{n-1} \). Thus

\[
b_N a_N > b_N a_N - b_n a_n \geq b_N + b_{N+1} + \cdots + b_n.
\]

However, it can be shown that \( \sum_{n>N} b_N \) diverges: in fact, since \( 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n \), we have \( b_n > 2^{-1-\ln n} = \frac{1}{2} n^{-\ln 2} > \frac{1}{2n} \), and we already know that \( \sum_{n>N} \frac{1}{2n} \) diverges.

Remark. As can be seen from both solutions, the value 2 in the problem can be increased to \( e \).
3. By the arithmetic–quadratic mean inequality, it suffices to prove that
\[
\frac{x_1^2}{(1 + x_1^2)^2} + \frac{x_2^2}{(1 + x_1^2 + x_2^2)^2} + \cdots + \frac{x_n^2}{(1 + x_1^2 + \cdots + x_n^2)^2} < 1.
\]
Observe that for \( k \geq 2 \) the following holds:
\[
\frac{x_k^2}{(1 + x_1^2 + \cdots + x_k^2)^2} \leq \frac{x_k^2}{(1 + \cdots + x_{k-1}^2)(1 + \cdots + x_k^2)} = \frac{1}{1 + x_1^2 + \cdots + x_{k-1}^2} - \frac{1}{1 + x_1^2 + \cdots + x_k^2}.
\]
For \( k = 1 \) we have \( \frac{x_1^2}{(1 + x_1^2)^2} \leq 1 - \frac{1}{1 + x_1^2} \). Summing these inequalities, we obtain
\[
\frac{x_1^2}{(1 + x_1^2)^2} + \cdots + \frac{x_n^2}{(1 + x_1^2 + \cdots + x_n^2)^2} \leq 1 - \frac{1}{1 + x_1^2 + \cdots + x_n^2} < 1.
\]

Second solution. Let \( a_n(k) = \sup \left( \frac{x_1}{k + x_1^2} + \cdots + \frac{x_n}{k^2 + x_1^2 + \cdots + x_n^2} \right) \) and \( a_n = a_n(1) \). We must show that \( a_n < \sqrt{n} \). Replacing \( x_i \) by \( kx_i \) shows that \( a_n(k) = a_n/k \). Hence
\[
a_n = \sup_{x_1} \left( \frac{x_1}{1 + x_1^2} + \frac{a_n-1}{\sqrt{1 + x_1^2}} \right) = \sup_{\theta} (\sin \theta \cos \theta + a_n-1 \cos \theta),
\]
where \( \tan \theta = x_1 \). The above supremum can be computed explicitly:
\[
a_n = \frac{1}{8} \sqrt{2} \left( 3(a_n-1 + \sqrt{a_n-1 + 8}) \right) \sqrt{4 - a_n-1 + a_n-1 \sqrt{a_n-1 + 8}}.
\]
However, the required inequality is weaker and can be proved more easily: if \( a_{n-1} < \sqrt{n} - 1 \), then by (1) \( a_n < \sin \theta + \sqrt{n} - 1 \cos \theta = \sqrt{n} \sin(\theta + \alpha) \leq \sqrt{n} \), for \( \alpha \in (0, \pi/2) \) with \( \tan \alpha = \sqrt{n} \).

4. Let (*) denote the given functional equation. Substituting \( y = 1 \) we get \( f(x)^2 = xf(x)f(1) \). If \( f(1) = 0 \), then \( f(x) = 0 \) for all \( x \), which is the trivial solution. Suppose \( f(1) = C \neq 0 \). Let \( G = \{y \in \mathbb{R} \mid f(y) \neq 0\} \). Then
\[
f(x) = \begin{cases} 
Cx & \text{if } x \in G, \\
0 & \text{otherwise.}
\end{cases}
\]
We must determine the structure of \( G \) so that the function defined by (1) satisfies (*).

(1) Clearly \( 1 \in G \), because \( f(1) \neq 0 \).
(2) If \( x \in G \), \( y \notin G \), then by (*) it holds \( f(xy)f(x) = 0 \), so \( xy \notin G \).
(3) If \( x, y \in G \), then \( x/y \in G \) (otherwise by \( 2^\circ \), \( y(x/y) = x \notin G \)).
(4) If \( x, y \in G \), then by \( 2^2 \) we have \( x^{-1} \in G \), so \( xy = y/x^{-1} \in G \).
Hence \( G \) is a set that contains 1, does not contain 0, and is closed under multiplication and division. Conversely, it is easy to verify that every such \( G \) in (1) gives a function satisfying (*).

5. Let \( a_1, a_2, \ldots, a_n \) satisfy the conditions of the problem. Then \( a_k > a_{k-1} \), and hence \( a_k \geq 2 \) for \( k = 1, \ldots, n \). The inequality \((a_{k+1} - 1)a_{k-1} \geq a_k^2(a_k - 1)\) can be rewritten as
\[
\frac{a_{k-1}}{a_k} + \frac{a_k}{a_{k+1} - 1} \leq \frac{a_{k-1}}{a_k - 1}.
\]
Summing these inequalities for \( k = i + 1, \ldots, n - 1 \) and using the obvious inequality \( \frac{a_{i-1}}{a_i} < \frac{a_i}{a_{i+1}} \), we obtain \( \frac{a_1}{a_2} + \cdots + \frac{a_{n-1}}{a_n} < \frac{a_1}{a_2 - 1} \). Therefore
\[
\frac{a_i}{a_{i+1}} \leq \frac{99}{100} - \frac{a_0}{a_1} - \cdots - \frac{a_{i-1}}{a_i} < \frac{a_i}{a_{i+1} - 1} \quad \text{for} \quad i = 1, 2, \ldots, n - 1. \tag{1}
\]
Consequently, given \( a_0, a_1, \ldots, a_i \), there is at most one possibility for \( a_{i+1} \).
In our case, (1) yields \( a_1 = 2, a_2 = 5, a_3 = 56, a_4 = 280^2 = 78400 \).
These values satisfy the conditions of the problem, so that this is a unique solution.

6. We shall determine a constant \( k > 0 \) such that
\[
\frac{a}{\sqrt{a^2 + 8bc}} \geq \frac{a^k}{a^k + b^k + c^k} \quad \text{for all} \quad a, b, c > 0. \tag{1}
\]
This inequality is equivalent to \((a^k + b^k + c^k)^2 \geq a^{2k-2}(a^2 + 8bc)\), which further reduces to
\[
(a^k + b^k + c^k)^2 - a^{2k} \geq 8a^{2k-2}bc.
\]
On the other hand, the AM–GM inequality yields
\[
(a^k + b^k + c^k)^2 - a^{2k} = (b^k + c^k)(2a^k + b^k + c^k) \geq 8a^{k/2}b^{3k/4}c^{3k/4},
\]
and therefore \( k = 4/3 \) is a good choice. Now we have
\[
\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq \frac{a^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}} + \frac{b^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}} + \frac{c^{4/3}}{a^{4/3} + b^{4/3} + c^{4/3}} = 1.
\]
Second solution. The numbers \( x = \frac{a}{\sqrt{a^2 + 8bc}}, y = \frac{b}{\sqrt{b^2 + 8ca}} \) and \( z = \frac{c}{\sqrt{c^2 + 8ab}} \) satisfy
\[
f(x, y, z) = \left( \frac{1}{x^2} - 1 \right) \left( \frac{1}{y^2} - 1 \right) \left( \frac{1}{z^2} - 1 \right) = 8^3.
\]
Our task is to prove \( x + y + z \geq 1 \).
Since \( f \) is decreasing on each of the variables \( x, y, z \), this is the same as proving that \( x, y, z > 0, x + y + z = 1 \) implies \( f(x, y, z) \geq 8^3 \). However, since \( \frac{1}{x} - 1 = \frac{(x+y+z)^2-x^2}{x^2} = \frac{(2x+y+z)(y+z)}{x^2} \), the inequality \( f(x, y, z) \geq 8^3 \)
becomes
\[
\frac{(2x + y + z)(x + 2y + z)(x + y + 2z)(y + z)(z + x)(x + y)}{x^2y^2z^2} \geq 8^3,
\]
which follows immediately by the AM–GM inequality.

**Third solution.** We shall prove a more general fact: the inequality
\[
\frac{a}{\sqrt{a^2+abc}} + \frac{b}{\sqrt{b^2+abc}} + \frac{c}{\sqrt{c^2+abc}} \geq \frac{3}{\sqrt{1+k}} \]
is true for all \( a, b, c > 0 \) if and only if \( k \geq 8 \).

Firstly suppose that \( k \geq 8 \). Setting \( x = bc/a^2, y = ca/b^2, z = ab/c^2 \), we reduce the desired inequality to
\[
F(x, y, z) = f(x) + f(y) + f(z) \geq \frac{3}{\sqrt{1+k}}, \quad \text{where } f(t) = \frac{1}{\sqrt{1+kt}}, \quad (2)
\]
for \( x, y, z > 0 \) such that \( xyz = 1 \). We shall prove (2) using the method of Lagrange multipliers.

The boundary of the set \( D = \{ (x, y, z) \in \mathbb{R}^3_+ \mid xyz = 1 \} \) consists of points \((x, y, z)\) with one of \(x, y, z\) being 0 and another one being \(+\infty\). If w.l.o.g. \( x = 0 \), then \( F(x, y, z) \geq f(x) = 1 \geq 3/\sqrt{1+k} \).

Suppose now that \((x, y, z)\) is a point of local minimum of \( F \) on \( D \). There exists \( \lambda \in \mathbb{R} \) such that \((x, y, z)\) is stationary point of the function \( F'(x, y, z) + \lambda xyz \). Then \((x, y, z, \lambda)\) is a solution to the system \( f'(x) + \lambda yz = f'(y) + \lambda xz = f'(z) + \lambda xy = 0 \). Eliminating \( \lambda \) gives us
\[
x f'(x) = y f'(y) = z f'(z), \quad xyz = 1. \quad (3)
\]
The function \( tf'(t) = \frac{-kt}{2(1+kt)^{3/2}} \) decreases on the interval \((0, 2/k] \) and increases on \([2/k, +\infty)\) because \((tf'(t))' = \frac{k(k-2)}{4(1+kt)^{5/2}} \). It follows that two of the numbers \(x, y, z\) are equal. If \( x = y = z \), then \((1, 1, 1)\) is the only solution to (3). Suppose that \( x = y \neq z \). Since \((y f'(y))^2 = (z f'(z))^2 = k^2(z-y)(k^3y^2z^2-3k^2yz-y-z)/4(k^3y^2z^2-3k^2yz-y-z)\), (3) gives us \( y^2z = 1 \) and \( k^3y^2z^2-3k^2yz-y-z = 0 \). Eliminating \( z \) we obtain an equation in \( y \), \( k^3/y^2 - 3k/y - 1/y^2 = 0 \), whose only real solution is \( y = k - 1 \). Thus \((k-1, k-1, 1/(k-1)^2)\) and the cyclic permutations are the only solutions to (3) with \( x, y, z \) being not all equal. Since \( F(k-1, k-1, 1/(k-1)^2) = (k+1)/\sqrt{k^2-1} \geq F(1, 1, 1) = 1 \), the inequality (2) follows.

For \( 0 < k < 8 \) we have that \( a/\sqrt{a^2+abc} + b/\sqrt{b^2+abc} + c/\sqrt{c^2+abc} \geq a/\sqrt{a^2+8bc} + b/\sqrt{b^2+8ca} + c/\sqrt{c^2+8ab} \geq 1 \). If we fix \( c \) and let \( a, b \) tend to 0, the first two summands will tend to 0 while the third will tend to 1. Hence the inequality cannot be improved.
7. It is evident that arranging of $A$ in increasing order does not diminish $m$. Thus we can assume that $A$ is nondecreasing. Assume w.l.o.g. that $a_1 = 1$, and let $b_i$ be the number of elements of $A$ that are equal to $i$ ($1 \leq i \leq n = a_{2001}$). Then we have $b_1 + b_2 + \cdots + b_n = 2001$ and

$$m = b_1b_2b_3 + b_2b_3b_4 + \cdots + b_{n-2}b_{n-1}b_n. \quad (1)$$

Now if $b_i, b_j$ ($i < j$) are two largest $b$’s, we deduce from (1) and the AM–GM inequality that $m \leq b_ib_j(b_1 + \cdots + b_{i-1} + b_{i+1} + \cdots + b_{j-1} + b_{j+1} + b_n) \leq \left(\frac{2001}{3}\right)^3 = 667^3 (b_1b_2b_3 \leq b_1b_ib_j$, etc.). The value $667^3$ is attained for

$b_1 = b_2 = 667$ (i.e., $a_1 = \cdots = a_{667} = 1$, $a_{668} = \cdots = a_{1334} = 2$, $a_{1335} = \cdots = a_{2001} = 3$). Hence the maximum of $m$ is $667^3$.

8. Suppose to the contrary that all the $S(a)$’s are different modulo $n!$. Then the sum of $S(a)$’s over all permutations $a$ satisfies $\sum_a S(a) \equiv 0 + 1 + \cdots + (n! - 1) = \frac{(n!-1)n!}{2} \equiv \frac{n!}{2} \pmod{n!}$. On the other hand, the coefficient of $c_i$ in $\sum_a S(a)$ is equal to $(n-1)!(1 + 2 + \cdots + n) = \frac{n+1}{2}n!$ for all $i$, from which we obtain

$$\sum_a S(a) \equiv \frac{n+1}{2}(c_1 + \cdots + c_n)n! \equiv 0 \pmod{n!}$$

for odd $n$. This is a contradiction.

9. Consider one such party. The result is trivially true if there is only one 3-clique, so suppose there exist at least two 3-cliques $C_1$ and $C_2$. We distinguish two cases:

(i) $C_1 = \{a, b, c\}$ and $C_2 = \{a, d, e\}$ for some distinct people $a, b, c, d, e$. If the departure of $a$ destroys all 3-cliques, then we are done. Otherwise, there is a third 3-clique $C_3$, which has a person in common with each of $C_1, C_2$ and does not include $a$: say, $C_3 = \{b, d, f\}$ for some $f$. We thus obtain another 3-clique $C_4 = \{a, b, d\}$, which has two persons in common with $C_3$, and the case (ii) is applied.

(ii) $C_1 = \{a, b, c\}$ and $C_2 = \{a, b, d\}$ for distinct people $a, b, c, d$. If the departure of $a, b$ leaves no 3-clique, then we are done. Otherwise, for some $e$ there is a clique $\{c, d, e\}$.

We claim that then the departure of $c, d$ breaks all 3-cliques. Suppose the opposite, that a 3-clique $C$ remains. Since $C$ shares a person with each of the 3-cliques $\{c, d, a\}, \{c, d, b\}, \{c, d, e\}$, it must be $C = \{a, b, e\}$. However, then $\{a, b, c, d, e\}$ is a 5-clique, which is assumed to be impossible.

10. For convenience let us write $a = 1776, b = 2001, 0 < a < b$. There are two types of historic sets:

(1) $\{x, x+a, x+a+b\}$ and (2) $\{x, x+b, x+a+b\}$.

We construct a sequence of historic sets $H_1, H_2, H_3, \ldots$ inductively as follows:
(i) \( H_1 = \{0, a, a + b\} \), and
(ii) Let \( y_n \) be the least nonnegative integer not occurring in \( U_n = H_1 \cap \cdots \cap H_n \). We take \( H_{n+1} \) to be \( \{y_n, y_n + a, y_n + a + b\} \) if \( y_n + a \not\in U_n \), and \( \{y_n, y_n + b, y_n + a + b\} \) otherwise.

It remains to show that this construction never fails. Suppose that it failed at the construction of \( H_{n+1} \). The element \( y_n + a + b \) is not contained in \( U_n \), since by the construction the smallest elements of \( H_1, \ldots, H_n \) are all less than \( y_n \). Hence the reason for the failure must be the fact that both \( y_n + a \) and \( y_n + b \) are covered by \( U_n \). Further, \( y_n + b \) must have been the largest element of its set \( H_k \), so the smallest element of \( H_k \) equals \( y_n - a \). But since \( y_n \) is not the least nonnegative integer not occurring in \( U_n \), we conclude that \( H_k \) is of type (2). This is a contradiction, because \( y_n \) was free, so by the algorithm we had to choose for \( H_k \) the set of type (1) (that is, \( \{y_n - a, y_n, y_n + b\} \)) first.

11. Let \((x_0, x_1, \ldots, x_n)\) be any such sequence: its terms are clearly nonnegative integers. Also, \( x_0 = 0 \) yields a contradiction, so \( x_0 > 0 \). Let \( m \) be the number of positive terms among \( x_1, \ldots, x_n \). Since \( x_i \) counts the terms equal to \( i \), the sum \( x_1 + \cdots + x_n \) counts the total number of positive terms in the sequence, which is known to be \( m + 1 \). Therefore among \( x_1, \ldots, x_n \) exactly \( m - 1 \) terms are equal to 1, one is equal to 2, and the others are 0. Only \( x_0 \) can exceed 2, and consequently at most one of \( x_3, x_4, \ldots \) can be positive. It follows that \( m \leq 3 \).

(i) \( m = 1 \): Then \( x_2 = 2 \) (since \( x_1 = 2 \) is impossible), so \( x_0 = 2 \). The resulting sequence is \((2, 0, 2, 0)\).

(ii) \( m = 2 \): Either \( x_1 = 2 \) or \( x_2 = 2 \). These cases yield \((1, 2, 1, 0)\) and \((2, 1, 2, 0, 0)\) respectively.

(iii) \( m = 3 \): This means that \( x_k > 0 \) for some \( k > 2 \). Hence \( x_0 = k \) and \( x_k = 1 \). Further, \( x_1 = 1 \) is impossible, so \( x_1 = 2 \) and \( x_2 = 1 \); there are no more positive terms in the sequence. The resulting sequence is \( (p, 2, 1, 0, \ldots, 0, 1, 0, 0, 0) \).

12. For each balanced sequence \( a = (a_1, a_2, \ldots, a_{2n}) \) denote by \( f(a) \) the sum of \( j \)'s for which \( a_j = 1 \) (for example, \( f(100101) = 1 + 4 + 6 = 11 \)). Partition the \( (2n) \) balanced sequences into \( n + 1 \) classes according to the residue of \( f \) modulo \( n + 1 \). Now take \( S \) to be a class of minimum size. Obviously \( |S| \leq \frac{1}{n+1} (2^n) \). We claim that every balanced sequence \( a \) is either a member of \( S \) or a neighbor of a member of \( S \). We consider two cases.

(i) Let \( a_1 \) be 1. It is easy to see that moving this 1 just to the right of the \( k \)th 0, we obtain a neighboring balanced sequence \( b \) with \( f(b) = f(a) + k \). Thus if \( a \not\in S \), taking a suitable \( k \in \{1, 2, \ldots, n\} \) we can achieve that \( b \in S \).

(ii) Let \( a_1 \) be 0. Taking this 0 just to the right of the \( k \)th 1 gives a neighbor \( b \) with \( f(b) = f(a) - k \), and the conclusion is similar to that of (i). This justifies our claim.
13. At any moment, let $p_i$ be the number of pebbles in the $i$th column, $i = 1, 2, \ldots$. The final configuration has obvious properties $p_1 \geq p_2 \geq \cdots$ and $p_{i+1} \in \{p_i, p_i - 1\}$. We claim that $p_{i+1} = p_i > 0$ is possible for at most one $i$.

Assume the opposite. Then the final configuration has the property that for some $r$ and $s > r$ we have $p_{r+1} = p_r$, $p_{s+1} = p_s > 0$ and $p_{r+k} = p_{r+1} - k + 1$ for all $k = 1, \ldots, s - r$. Consider the earliest configuration, say $C$, with this property. What was the last move before $C$? The only possibilities are moving a pebble either from the $r$th or from the $s$th column; however, in both cases the configuration preceding this last move had the same property, contradicting the assumption that $C$ is the earliest. Therefore the final configuration looks as follows: $p_1 = a \in \mathbb{N}$, and for some $r$, $p_i$ equals $a - (i - 1)$ if $i \leq r$, and $a - (i - 2)$ otherwise. It is easy to determine $a, r$: since $n = p_1 + p_2 + \cdots = \frac{(a+1)(a+2)}{2} - r$, we get $\frac{a(a+1)}{2} \leq n < \frac{(a+1)(a+2)}{2}$, from which we uniquely find $a$ and then $r$ as well.

The final configuration for $n = 13$:

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14. We say that a problem is difficult for boys if at most two boys solved it, and difficult for girls if at most two girls solved it.

Let us estimate the number of pairs boy-girl both of whom solved some problem difficult for boys. Consider any girl. By the condition (ii), among the six problems she solved, at least one was solved by at least 3 boys, and hence at most 5 were difficult for boys. Since each of these problems was solved by at most 2 boys and there are 21 girls, the considered number of pairs does not exceed $5 \cdot 2 \cdot 21 = 210$.

Similarly, there are at most 210 pairs boy-girl both of whom solved some problem difficult for girls. On the other hand, there are $21^2 > 2 \cdot 210$ pairs boy-girl, and each of them solved one problem in common. Thus some problems were difficult neither for girls nor for boys, as claimed.

Remark. The statement can be generalized: if $2(m - 1)(n - 1) + 1$ boys and as many girls participated, and nobody solved more than $m$ problems, then some problem was solved by at least $n$ boys and $n$ girls.

15. Let $MNPQ$ be the square inscribed in $\triangle ABC$ with $M \in AB$, $N \in AC$, $P, Q \in BC$, and let $AA_1$ meet $MN, PQ$ at $K, X$ respectively. Put $MK = PX = m$, $NK = QX = n$, and $MN = d$. Then

\[
\frac{BX}{XC} = \frac{m}{n} = \frac{BX + m}{XC + n} = \frac{BP}{CQ} = \frac{d \cot \beta + d}{d \cot \gamma + d} = \frac{\cot \beta + 1}{\cot \gamma + 1}.
\]

Similarly, if $BB_1$ and $CC_1$ meet $AC$ and $BC$ at $Y, Z$ respectively then

\[
\frac{CY}{YA} = \frac{\cot \gamma + 1}{\cot \alpha + 1} \quad \text{and} \quad \frac{AZ}{ZB} = \frac{\cot \beta + 1}{\cot \alpha + 1}.
\]

Therefore $\frac{BX}{XC} \frac{CY}{YA} \frac{AZ}{ZB} = 1$, so by Ceva’s theorem, $AX, BY, CZ$ have a common point.
Second solution. Let $A_2$ be the center of the square constructed over $BC$ outside $\triangle ABC$. Since this square and the inscribed square corresponding to the side $BC$ are homothetic, $A, A_1,$ and $A_2$ are collinear. Points $B_2, C_2$ are analogously defined. Denote the angles $BAA_2, A_2AC, CBB_2, B_2BA, ACC_2, C_2CB$ by $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$. By the law of sines we have
\[
\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{\sin(\beta + 45^\circ)}{\sin(\gamma + 45^\circ)}, \quad \frac{\sin \beta_1}{\sin \beta_2} = \frac{\sin(\gamma + 45^\circ)}{\sin(\alpha + 45^\circ)}, \quad \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{\sin(\alpha + 45^\circ)}{\sin(\beta + 45^\circ)}.
\]
Since the product of these ratios is 1, by the trigonometric Ceva’s theorem $AA_2, BB_2, CC_2$ are concurrent.

16. Since $\angle OCP = 90^\circ - \angle A$, we are led to showing that $\angle OCP > \angle COP$, i.e., $OP > CP$. By the triangle inequality it suffices to prove $CP < \frac{1}{2} CO$. Let $CO = R$. The law of sines yields $CP = AC \cos \gamma = 2R \sin \beta \cos \gamma < 2R \sin \beta \cos(\beta + 30^\circ)$. Finally, we have
\[
2 \sin \beta \cos(\beta + 30^\circ) = \sin(2\beta + 30^\circ) - \sin 30^\circ \leq \frac{1}{2},
\]
which completes the proof.

17. Let us investigate a more general problem, in which $G$ is any point of the plane such that $AG, BG, CG$ are sides of a triangle. Let $F$ be the point in the plane such that $BC : CF : FB = AG : BG : CG$ and $F, A$ lie on different sides of $BC$. Then by Ptolemy’s inequality, on $BPCF$ we have $AG \cdot AP + BG \cdot BP + CG \cdot CP = AG \cdot AP + \frac{AG}{BC}(CF \cdot BP + BF \cdot CP) \geq AG \cdot AP + \frac{AG}{BC} BC \cdot PF$. Hence
\[
AG \cdot AP + BG \cdot BP + CG \cdot CP \geq AG \cdot AF,
\]
where equality holds if and only if $P$ lies on the segment $AF$ and on the circle $BCF$. Now we return to the case of $G$ the centroid of $\triangle ABC$. We claim that $F$ is then the point $\hat{G}$ in which the line $AG$ meets again the circumcircle of $\triangle BGC$. Indeed, if $M$ is the midpoint of $AB$, by the law of sines we have $\frac{BC}{CG} = \frac{\sin \angle B\hat{G}C}{\sin \angle C\hat{G}B} = \frac{\angle BGM}{\angle AMG} = \frac{AG}{BG}$, and similarly $\frac{BC}{BG} = \frac{AG}{CG}$. Thus (1) implies
\[
AG \cdot AP + BG \cdot BP + CG \cdot CP \geq AG \cdot A\hat{G}.
\]
It is easily seen from the above considerations that equality holds if and only if $P \equiv G$, and then the (minimum) value of $AG \cdot AP + BG \cdot BP + CG \cdot CP$ equals
\[ AG^2 + BG^2 + CG^2 = \frac{a^2 + b^2 + c^2}{3}. \]

**Second solution.** Notice that \( AG \cdot AP \geq \overrightarrow{AG} \cdot \overrightarrow{AP} = \overrightarrow{AG} \cdot (\overrightarrow{AG} + \overrightarrow{PG}). \) Summing this inequality with analogous inequalities for \( BG \cdot BP \) and \( CG \cdot CP \) gives us \( AG \cdot AP + BG \cdot BP + CG \cdot CP \geq AG^2 + BG^2 + CG^2 + (\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG}) \cdot \overrightarrow{PG} = AG^2 + BG^2 + CG^2 = \frac{a^2 + b^2 + c^2}{3}. \) Equality holds if and only if \( P \equiv Q. \)

18. Let \( \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \) denote the angles \( \angle MAB, \angle MBA, \angle BMA, \angle MAC, \angle MBA, \angle MCB \) respectively. Then \( \frac{MC \cdot MA'}{MB^2} = \sin \alpha_1 \sin \alpha_2, \frac{MA \cdot MB'}{MC^2} = \sin \beta_1 \sin \beta_2, \frac{MC \cdot MA'}{MB^2} = \sin \gamma_1 \sin \gamma_2; \) hence
\[ p(M)^2 = \sin \alpha_1 \sin \alpha_2 \sin \beta_1 \sin \beta_2 \sin \gamma_1 \sin \gamma_2. \]

Since \( \sin \alpha_1 \sin \alpha_2 = \frac{1}{2}(\cos(\alpha_1 - \alpha_2) - \cos(\alpha_1 + \alpha_2)) \leq \frac{1}{2}(1 - \cos \alpha) = \sin^2 \frac{\alpha}{2}, \) we conclude that
\[ p(M) \leq \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}. \]

Equality occurs when \( \alpha_1 = \alpha_2, \beta_1 = \beta_2, \) and \( \gamma_1 = \gamma_2, \) that is, when \( M \) is the incenter of \( \triangle ABC. \)

It is well known that \( \mu(ABC) = \frac{\alpha}{2} \sin \beta \sin \gamma \) is maximal when \( \triangle ABC \) is equilateral (it follows, for example, from Jensen’s inequality applied to \( \ln \sin x \)). Hence \( \max \mu(ABC) = \frac{1}{8}. \)

19. It is easy to see that the hexagon \( AEBFCD \) is convex and \( \angle AEB + \angle BFC + \angle CDA = 360^\circ. \) Using this relation we obtain that the circles \( \omega_1, \omega_2, \omega_3 \) with centers at \( D, E, F \) and radii \( DA, EB, FC \) respectively all pass through a common point \( O. \) Indeed, if \( \omega_1 \cap \omega_2 = \{O\}, \) then \( \angle AOB = 180^\circ - \angle AEB / 2 \) and \( \angle BOC = 180^\circ - \angle BFC / 2; \) hence \( \angle COA = 180^\circ - \angle CDA / 2 \) as well, i.e., \( O \in \omega_3. \) The point \( O \) is the reflection of \( A \) with respect to \( DE. \) Similarly, it is also the reflection of \( B \) with respect to \( EF, \) and that of \( C \) with respect to \( FD. \) Hence
\[ \frac{DB}{DD'} = 1 + \frac{D'B}{DD'} = 1 + \frac{S_{EBF}}{S_{EDF}} = 1 + \frac{S_{ODEF}}{S_{DEF}}. \]

Analogously \( \frac{EC}{EE'} = 1 + \frac{S_{ODEF}}{S_{DEF}} \) and \( \frac{FA}{FF'} = 1 + \frac{S_{ODEF}}{S_{DEF}}. \) Adding these relations gives us
\[
\frac{DB}{DD'} + \frac{EC}{EE'} + \frac{FA}{FF'} = 3 + \frac{S_{OEF} + S_{ODF} + S_{ODE}}{S_{DEF}} = 4.
\]

20. By Ceva’s theorem, we can choose real numbers \(x, y, z\) such that
\[
\frac{BD}{DC} = \frac{z}{y}, \quad \frac{CE}{EA} = \frac{x}{z}, \quad \text{and} \quad \frac{AF}{FB} = \frac{y}{x}.
\]
The point \(P\) lies outside the triangle \(ABC\) if and only if \(x, y, z\) are not all of the same sign. In what follows, \(S_X\) will denote the signed area of a figure \(X\).

Let us assume that the area \(S_{ABC}\) of \(\triangle ABC\) is 1. Since \(S_{PBC} : S_{PCA} : S_{PAB} = x : y : z\) and \(S_{PBD} : S_{PDC} = z : y\), it follows that
\[
S_{PBD} = \frac{z}{y + z} \cdot \frac{y}{x + y + z}, \quad S_{PCE} = \frac{1}{y(y + 1)} \cdot \frac{xy}{x + y + z}, \quad S_{PAF} = \frac{1}{x + y + z} \cdot \frac{xy}{x + y + z}.
\]
By the condition of the problem we have \(|S_{PBD}| = |S_{PCE}| = |S_{PAF}|\), or
\[
|x(x + y)| = |y(y + z)| = |z(z + x)|.
\]

Obviously \(x, y, z\) are nonzero, so that we can put w.l.o.g. \(z = 1\). At least two of the numbers \(x(x + y), y(y + 1), 1(1 + x)\) are equal, so we can assume that \(x(x + y) = y(y + 1)\). We distinguish two cases:

(i) \(x(x + y) = y(y + 1) = 1 + x\). Then \(x = y^2 + y - 1\), from which we obtain \((y^2 + y - 1)(y^2 + 2y - 1) = y(y + 1)\). Simplification gives
\[
y^4 + 3y^3 - y^2 - 4y + 1 = 0,
\]
or
\[
(y - 1)(y^3 + 4y^2 + 3y - 1) = 0.
\]
If \(y = 1\), then also \(z = x = 1\), so \(P\) is the centroid of \(\triangle ABC\), which is not an exterior point. Hence \(y^3 + 4y^2 + 3y - 1 = 0\). Now the signed area of each of the triangles \(PBD, PCE, PAF\) equals
\[
S_{PAF} = \frac{yz}{(x + y)(x + y + z)} = \frac{1}{(y^2 + 2y - 1)(y^2 + 2y)} = \frac{1}{y^3 + 4y^2 + 3y - 2} = -1.
\]

It is easy to check that not both of \(x, y\) are positive, implying that \(P\) is indeed outside \(\triangle ABC\). This is the desired result.

(ii) \(x(x + y) = y(y + 1) = -1 - x\). In this case we are led to
\[
f(y) = y^4 + 3y^3 + y^2 - 2y + 1 = 0.
\]
We claim that this equation has no real solutions. In fact, assume that \(y_0\) is a real root of \(f(y)\). We must have \(y_0 < 0\), and hence \(u = -y_0 > 0\) satisfies
\[
3u^3 - u^4 = (u + 1)^2.
\]
On the other hand,
\[3u^3 - u^4 = u^3(3 - u) = 4u \left( \frac{u}{2} \right) \left( \frac{u}{2} \right) (3 - u)\]

\[\leq 4u \left( \frac{u/2 + u/2 + 3 - u}{3} \right)^3 = 4u\]

\[\leq (u + 1)^2,\]

where at least one of the inequalities is strict, a contradiction.

**Remark.** The official solution was incomplete, missing the case (ii).

21. We denote by \(p(XYZ)\) the perimeter of a triangle \(XYZ\).

If \(O\) is the circumcenter of \(\triangle ABC\), then \(A_1, B_1, C_1\) are the midpoints of the corresponding sides of the triangle, and hence \(p(A_1B_1C_1) = p(AB_1C_1) = p(A_1B_1C)\).

Conversely, suppose that \(p(A_1B_1C_1) \geq p(AB_1C_1), p(A_1B_1C), p(A_1B_1C), p(A_1B_1C_1)\). Let \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\) denote \(\angle B_1A_1C, \angle C_1A_1B, \angle C_1B_1A, \angle A_1B_1C, \angle A_1C_1B, \angle B_1C_1A\).

Suppose that \(\gamma_1, \beta_2 \geq \alpha\). If \(A_2\) is the fourth vertex of the parallelogram \(B_1A_2C_1A\), then these conditions imply that \(A_1\) is in the interior or on the border of \(\triangle B_1C_1A\), and therefore \(p(A_1B_1C_1) \leq p(A_2B_1C_1) = p(AB_1C_1)\). Moreover, if one of the inequalities \(\gamma_1 \geq \alpha, \beta_2 \geq \alpha\) is strict, then \(p(A_1B_1C_1)\) is strictly less than \(p(AB_1C_1)\), contrary to the assumption. Hence

\[\beta_2 \geq \alpha \implies \gamma_1 \leq \alpha,\]

\[\gamma_2 \geq \beta \implies \alpha_1 \leq \beta,\]

\[\alpha_2 \geq \gamma \implies \beta_1 \leq \gamma,\]

the last two inequalities being obtained analogously to the first one. Because of the symmetry, there is no loss of generality in assuming that \(\gamma_1 \leq \alpha\). Then since \(\gamma_1 + \alpha_2 = 180^\circ - \beta = \alpha + \gamma\), it follows that \(\alpha_2 \geq \gamma\).

From (1) we deduce \(\beta_1 \leq \gamma\), which further implies \(\gamma_2 \geq \beta\). Similarly, this leads to \(\alpha_1 \leq \beta\) and \(\beta_2 \geq \alpha\). To sum up,

\[\gamma_1 \leq \alpha \leq \beta_2, \quad \alpha_1 \leq \beta \leq \gamma_2, \quad \beta_1 \leq \gamma \leq \alpha_2.\]

Since \(OA_1BC_1\) and \(OB_1CA_1\) are cyclic, we have \(\angle A_1OB = \gamma_1\) and \(\angle A_1OC = \beta_2\). Hence \(BO : CO = \cos \beta_2 : \cos \gamma_1\), hence \(BO \leq CO\). Analogously, \(CO \leq AO\) and \(AO \leq BO\). Therefore \(AO = BO = CO\), i.e., \(O\) is the circumcenter of \(ABC\).

22. Let \(S\) and \(T\) respectively be the points on the extensions of \(AB\) and \(AQ\) over \(B\) and \(Q\) such that \(BS = BP\) and \(QT = QB\). It is given that \(AS = AB + BP = AQ + QB = AT\). Since \(\angle PAS = \angle PAT\), the triangles \(APS\)
and $\angle ATP = \angle ASP = \beta/2 = \angle QBP$. Hence $\angle QTP = \angle QBP$.

If $P$ does not lie on $BT$, then the last equality implies that $\triangle QBP$ and $\triangle QTP$ are congruent, so $P$ lies on the internal bisector of $\angle BQT$. But $P$ also lies on the internal bisector of $\angle QAB$; consequently, $P$ is an excenter of $\triangle QAB$, thus lying on the internal bisector of $\angle QBS$ as well. It follows that $\angle PBQ = \beta/2 = \angle PBS = 180^\circ - \beta$, so $\beta = 120^\circ$, which is impossible. Therefore $P \in BT$, which means that $T \equiv C$. Now from $QC = QB$ we conclude that $120^\circ - \beta = \gamma = \beta/2$, i.e., $\beta = 80^\circ$ and $\gamma = 40^\circ$.

23. For each positive integer $x$, define $\alpha(x) = x/10^r$ if $r$ is the positive integer satisfying $10^r \leq x < 10^{r+1}$. Observe that if $\alpha(x)\alpha(y) < 10$ for some $x, y \in \mathbb{N}$, then $\alpha(xy) = \alpha(x)\alpha(y)$. If, as usual, $[t]$ means the integer part of $t$, then $[\alpha(x)]$ is actually the leftmost digit of $x$.

Now suppose that $n$ is a positive integer such that $k \leq \alpha((n+k)!) < k+1$ for $k = 1, 2, \ldots, 9$. We have

$$1 < \alpha(n+k) = \frac{\alpha((n+k)!) - \alpha((n+k-1)!)}{\alpha((n+k-1)!)} < \frac{k+1}{k-1} \leq 3 < \frac{5}{4}$$

from which we obtain $\alpha(n+k+1) > \alpha(n+k)$ (the opposite can hold only if $\alpha(n+k) \geq 9$). Therefore

$$1 < \alpha(n+2) < \cdots < \alpha(n+9) \leq \frac{5}{4}.$$ 

On the other hand, this implies that $\alpha((n+4)!) = \alpha((n+1)!)*\alpha(n+2)*\alpha(n+3)*\alpha(n+4) < (5/4)^3\alpha((n+1)!)$ < 4, contradicting the assumption that the leftmost digit of $(n+4)!$ is 4.

24. We shall find the general solution to the system. Squaring both sides of the first equation and subtracting twice the second equation we obtain $(x-y)^2 = z^2 + u^2$. Thus $(z, u, x-y)$ is a Pythagorean triple. Then it is well known that there are positive integers $t, a, b$ such that $z = t(a^2 - b^2)$, $u = 2tab$ (or vice versa), and $x-y = t(a^2 + b^2)$. Using that $x+y = z+u$ we come to the general solution:

$$x = t(a^2 + ab), \quad y = t(ab - b^2); \quad z = t(a^2 - b^2), \quad u = 2tab.$$ 

Putting $a/b = k$ we obtain

$$\frac{x}{y} = \frac{k^2 + k}{k - 1} = 3 + (k - 1) + \frac{2}{k - 1} \geq 3 + 2\sqrt{2},$$

with equality for $k - 1 = \sqrt{2}$. On the other hand, $k$ can be arbitrarily close to $1 + \sqrt{2}$, and so $x/y$ can be arbitrarily close to $3 + 2\sqrt{2}$. Hence $m = 3 + 2\sqrt{2}$.

**Remark.** There are several other techniques for solving the given system. The exact lower bound of $m$ itself can be obtained as follows: by the system $\left(\frac{x}{y}\right)^2 - 6\frac{x}{y} + 1 = \left(\frac{z-u}{y}\right)^2 \geq 0$, so $x/y \geq 3 + 2\sqrt{2}$. 

25. Define $b_n = |a_{n+1} - a_n|$ for $n \geq 1$. From the equalities $a_{n+1} = b_{n-1} + b_{n-2}$, from $a_n = b_{n-2} + b_{n-3}$ we obtain $b_n = |b_{n-1} - b_{n-3}|$. From this relation we deduce that $b_m \leq \max(b_n, b_{n+1}, b_{n+2})$ for all $m \geq n$, and consequently $b_n$ is bounded.

Lemma. If $\max(b_n, b_{n+1}, b_{n+2}) = M \geq 2$, then $\max(b_{n+6}, b_{n+7}, b_{n+8}) \leq M - 1$.

Proof. Assume the opposite. Suppose that $b_j = M$, $j \in \{n, n+1, n+2\}$, and let $b_{j+1} = x$ and $b_{j+2} = y$. Thus $b_{j+3} = M - y$. If $x, y, M - y$ are all less than $M$, then the contradiction is immediate. The remaining cases are these:

(i) $x = M$. Then the sequence has the form $M, M, y, M - y, y, \ldots$, and since $\max(y, M - y, y, \ldots) = M$, we must have $y = 0$ or $y = M$.

(ii) $y = M$. Then the sequence has the form $M, x, M, 0, x, M - x, \ldots$, and since $\max(0, x, M - x) = M$, we must have $x = 0$ or $x = M$.

(iii) $y = 0$. Then the sequence is $M, x, 0, M - x, M - x, x, \ldots$, and since $\max(M - x, x, x) = M$, we have $x = 0$ or $x = M$.

In every case $M$ divides both $x$ and $y$. From the recurrence formula $M$ also divides $b_i$ for every $i < j$. However, $b_2 = 12^{12} - 11^{11}$ and $b_4 = 11^{11}$ are relatively prime, a contradiction.

From $\max(b_1, b_2, b_3) \leq 13^{13}$ and the lemma we deduce inductively that $b_n \leq 1$ for all $n \geq 6 \cdot 13^{13} - 5$. Hence $a_n = b_{n-2} + b_{n-3}$ takes only the values $0, 1, 2$ for $n \geq 6 \cdot 13^{13} - 2$. In particular, $a_{1414}$ is $0, 1, 2$. On the other hand, the sequence $a_n$ modulo 2 is as follows: $1, 0, 1, 0, 0, 1, 1, 0, 1, 0, \ldots$; and therefore it is periodic with period 7. Finally, $14^{14} \equiv 0 \pmod{7}$, from which we obtain $a_{1414} \equiv a_7 \equiv 1 \pmod{2}$. Therefore $a_{1414} = 1$.

26. Let $C$ be the set of those $a \in \{1, 2, \ldots, p-1\}$ for which $a^{p-1} \equiv 1 \pmod{p^2}$. At first, we observe that $a, p - a$ do not both belong to $C$, regardless of the value of $a$. Indeed, by the binomial formula,

$$(p-a)^{p-1} - a^{p-1} \equiv -(p-1)p a^{p-2} \not\equiv 0 \pmod{p^2}.$$ 

As a consequence we deduce that $|C| \leq \frac{p-1}{2}$. Further, we observe that $p - k \in C \iff k \equiv k(p-k)^{p-1} \pmod{p^2}$, i.e.,

$$p - k \in C \iff k \equiv k(k^{p-1} - (p-1)p k^{p-2}) \equiv k^p + p \pmod{p^2}. \quad (1)$$

Now assume the contrary to the claim, that for every $a = 1, \ldots, p - 2$ one of $a, a+1$ is in $C$. In this case it is not possible that $a, a+1$ are both in $C$, for then $p - a, p - a - 1 \not\in C$. Thus, since $1 \in C$, we inductively obtain that $2, 4, \ldots, p - 1 \not\in C$ and $1, 3, 5, \ldots, p - 2 \in C$. In particular, $p - 2, p - 4 \in C$, which is by (1) equivalent to $2 \equiv 2^p + p$ and $4 \equiv 4^p + p \pmod{p^2}$.

However, squaring the former equality and subtracting the latter, we obtain $2^{p+1}p \equiv p \pmod{p^2}$, or $4 \equiv 1 \pmod{p}$, which is a contradiction unless $p = 3$. This finishes the proof.
The given equality is equivalent to \( a^2 - ac + c^2 = b^2 + bd + d^2 \). Hence
\[
(ab + cd)(ad + bc) = ac(b^2 + bd + d^2) + bd(a^2 - ac + c^2),
\]
or equivalently,
\[
(ab + cd)(ad + bc) = (ac + bd)(a^2 - ac + c^2). 
\]
(1)

Now suppose that \( ab + cd \) is prime. It follows from \( a > b > c > d \) that
\[
ab + cd > ac + bd > ad + bc;
\]
hence \( ac + bd \) is relatively prime with \( ab + cd \). But then (1) implies that \( ac + bd \) divides \( ad + bc \), which is impossible by (2).

**Remark.** Alternatively, (1) could be obtained by applying the law of cosines and Ptolemy’s theorem on a quadrilateral \( XYZT \) with \( XY = a, YZ = c, ZT = b, TX = d \) and \( \angle Y = 60^\circ, \angle T = 120^\circ \).

28. Yes. The desired result is an immediate consequence of the following fact applied on \( p = 101 \).

**Lemma.** For any odd prime number \( p \), there exist \( p \) nonnegative integers less than \( 2p^2 \) with all pairwise sums mutually distinct.

**Proof.** We claim that the numbers \( a_n = 2np + (n^2) \) have the desired property, where \( (x) \) denotes the remainder of \( x \) upon division by \( p \).
Suppose that \( a_k + a_l = a_m + a_n \). By the construction of \( a_i \), we have
\[
2p(k+l) \leq a_k + a_l < 2p(k+l+1).
\]
Hence we must have \( k+l = m+n \), and therefore also \( (k^2) + (l^2) = (m^2) + (n^2) \). Thus
\[
k + l \equiv m + n \quad \text{and} \quad k^2 + l^2 \equiv m^2 + n^2 \pmod{p}.
\]

But then it holds that \( (k-l)^2 = 2(k^2 + l^2) - (k+l)^2 \equiv (m-n)^2 \pmod{p} \), so \( k - l \equiv \pm(m - n) \), which leads to \( (k, l) = (m, n) \). This proves the lemma.
4.43 Solutions to the Shortlisted Problems of IMO 2002

1. Consider the given equation modulo 9. Since each cube is congruent to either $-1$, $0$ or $1$, whereas $2002^{2002} \equiv 4^{2002} = 4 \cdot 64^{667} \equiv 4 \pmod{9}$, we conclude that $t \geq 4$.

On the other hand, $2002^{2002} = 2002 \cdot (2002^{667})^3 = (10^3 + 10^3 + 1^3)(2002^{667})^3$ is a solution with $t = 4$. Hence the answer is 4.

2. Set $S = d_1 d_2 + \cdots + d_k - 1 d_k$. Since $d_i/n = 1/d_{k+1-i}$, we have $S/n = \frac{1}{d_{k+1}} + \cdots + \frac{1}{d_2 d_1}$. Hence

$$\frac{1}{d_2 d_1} \leq S/n^2 \leq \left( \frac{1}{d_{k+1}} - \frac{1}{d_k} \right) + \cdots + \left( \frac{1}{d_1} - \frac{1}{d_2} \right) = 1 - \frac{1}{d_k} < 1,$$

or (since $d_1 = 1$) $1 < \frac{n^2}{S} \leq d_2$. This shows that $S < n^2$.

Also, if $S$ is a divisor of $n^2$, then $n^2/S$ is a nontrivial divisor of $n^2$ not exceeding $d_2$. But $d_2$ is obviously the least prime divisor of $n$ (and also of $n^2$), so we must have $n^2/S = d_2$, which holds if and only if $n$ is prime.

3. We observe that if $a$, $b$ are coprime odd numbers, then $\gcd(2^a + 1, 2^b + 1) = 3$. In fact, this g.c.d. divides $\gcd(2^{2a} - 1, 2^{2b} - 1) = 2^{\gcd(2a, 2b)} - 1 = 2^2 - 1 = 3$, while 3 obviously divides both $2^a + 1$ and $2^b + 1$. In particular, if $3 \nmid b$, then $3^3 \mid 2^b + 1$, so $2^a + 1$ and $(2^b + 1)/3$ are coprime; consequently $2^{ab} + 1$ (being divisible by $2^a + 1$, $2^b + 1$) is divisible by $(2^a + 1)(2^b + 1)/3$.

Now we prove the desired result by induction on $n$. For $n = 1$, $2^{p_n} + 1$ is divisible by 3 and exceeds 3, so it has at least 4 divisors. Assume that $2^a + 1 = 2^{p_1 \cdots p_n} + 1$ has at least $4^{n-1}$ divisors and consider $N = 2^{ab} + 1 = 2^{p_1 \cdots p_n} + 1$ (where $b = p_n$). As above, $2^a + 1$ and $2^{ab} + 1$ are coprime, and thus $Q = (2^a + 1)(2^{ab} + 1)/3$ has at least $2 \cdot 4^{n-1}$ divisors. Moreover, $N$ is divisible by $Q$ and is greater than $Q^2$ (indeed, $N > 2^{ab} > 2^{2a}2^{ab} > Q^2$ if $a,b \geq 5$). Then $N$ has at least twice as many divisors as $Q$ (because for every $d \mid Q$ both $d$ and $N/d$ are divisors of $N$), which counts up to $4^n$ divisors, as required.

Remark. With some knowledge of cyclotomic polynomials, one can show that $2^{p_1 \cdots p_n} + 1$ has at least $2^{2^{n-1}}$ divisors, far exceeding $4^n$.

4. For $a = b = c = 1$ we obtain $m = 12$. We claim that the given equation has infinitely many solutions in positive integers $a, b, c$ for this value of $m$.

After multiplication by $abc(a+b+c)$ the equation $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} - \frac{12}{a+b+c} = 0$ becomes

$$a^2(b + c) + b^2(c + a) + c^2(a + b) + a + b + c - 9abc = 0. \quad (1)$$

We must show that this equation has infinitely many solutions in positive integers. Suppose that $(a, b, c)$ is one such solution with $a < b < c$. Regarding $(1)$ as a quadratic equation in $a$, we see by Vieta’s formula that $(b, c, \frac{bc+1}{a})$ also satisfies $(1)$.
Define $(a_n)_{n=0}^\infty$ by $a_0 = a_1 = a_2 = 1$ and $a_{n+1} = \frac{a_na_{n-1}+1}{a_{n-2}}$ for each $n > 1$.

We show that all $a_n$'s are integers. This procedure is fairly standard. The above relation for $n$ and $n-1$ gives $a_{n+1}a_{n-2} = a_na_{n-1} + 1$ and $a_{n-1}a_{n-2} + 1 = a_na_{n-3}$, so that adding yields $a_{n-2}(a_{n-1} + a_{n+1}) = a_n(a_{n-1} + a_{n-3})$. Therefore $a_{n+1}a_{n-1} = a_{n-1}a_{n-3}$, and from which it follows that

$$\frac{a_{n+1} + a_{n-1}}{a_n} = \begin{cases} a_2 + a_n & 2 \text{ for } n \text{ odd;} \\ a_2 + a_1 & 3 \text{ for } n \text{ even.} \end{cases}$$

It is now an immediate consequence that every $a_n$ is integral. Also, this above consideration implies that $(a_{n-1}, a_n, a_{n+1})$ is a solution of (1) for each $n \geq 1$. Since $a_n$ is strictly increasing, this gives an infinity of solutions in integers.

Remark. There are infinitely many values of $m \in \mathbb{N}$ for which the given equation has at least one solution in integers, and each of those values admits an infinity of solutions.

5. Consider all possible sums $c_1a_1 + c_2a_2 + \cdots + c_na_n$, where each $c_i$ is an integer with $0 \leq c_i < m$. There are $m^n$ such sums, and if any two of them give the same remainder modulo $m^n$, say $\sum c_ia_i \equiv \sum d_ia_i \pmod{m^n}$, then $\sum (c_i - d_i)a_i$ is divisible by $m^n$, and since $|c_i - d_i| < m$, we are done. We claim that two such sums must exist.

Suppose to the contrary that the sums $\sum c_ia_i$ ($0 \leq c_i < m$) give all the different remainders modulo $m^n$. Consider the polynomial

$$P(x) = \sum x^{c_1a_1 + \cdots + c_na_n},$$

where the sum is taken over all $(c_1, \ldots, c_n)$ with $0 \leq c_i < m$. If $\xi$ is a primitive $m^n$th root of unity, then by the assumption we have

$$P(\xi) = 1 + \xi + \cdots + \xi^{m^n-1} = 0.$$

On the other hand, $P(x)$ can be factored as

$$P(x) = \prod_{i=1}^n (1 + x^{a_1} + \cdots + x^{(m-1)a_i}) = \prod_{i=1}^n \frac{1 - x^{m_a_i}}{1 - x^{a_i}},$$

so that none of its factors is zero at $x = \xi$ because $ma_i$ is not divisible by $m^n$. This is obviously a contradiction.

Remark. The example $a_i = m^{i-1}$ for $i = 1, \ldots, n$ shows that the condition that no $a_i$ is a multiple of $m^{n-1}$ cannot be removed.

6. Suppose that $(m, n)$ is such a pair. Assume that division of the polynomial $F(x) = x^n + x - 1$ by $G(x) = x^n + x^2 - 1$ gives the quotient $Q(x)$ and remainder $R(x)$. Since $\deg R(x) < \deg G(x)$, for $x$ large enough $|R(x)| < |G(x)|$; however, $R(x)$ is divisible by $G(x)$ for infinitely many integers $x$, so
it is equal to zero infinitely often. Hence \( R \equiv 0 \), and thus \( F(x) \) is exactly divisible by \( G(x) \).

The polynomial \( G(x) \) has a root \( \alpha \) in the interval \((0, 1)\), because \( G(0) = -1 \) and \( G(1) = 1 \). Then also \( F(\alpha) = 0 \), so that

\[
a^m + \alpha = a^n + \alpha^2 = 1.
\]

If \( m \geq 2n \), then \( 1 - \alpha = a^m \leq (a^n)^2 = (1 - \alpha^2)^2 \), which is equivalent to \( \alpha^2 + \alpha - 1 > a^m + \alpha - 1 = 0 \); hence \( m < 2n \).

Now we have \( F(x)/G(x) = x^{m-n} - (x^{m-n+2} - x^{m-n} - x + 1)/G(x) \), so \( H(x) = x^{m-n+2} - x^{m-n} - x + 1 \) is also divisible by \( G(x) \); but \( \deg H(x) = m - n + 2 \leq n + 1 = \deg G(x) + 1 \), from which we deduce that either \( H(x) = G(x) \) or \( H(x) = (x - a)G(x) \) for some \( a \in \mathbb{Z} \). The former case is impossible. In the latter case we must have \( m = 2n - 1 \), and thus \( H(x) = x^{n+1} - x^{n-1} - x + 1 \); on the other hand, putting \( x = 1 \) gives \( a = 1 \), so \( H(x) = (x - 1)(x^n + x^2 - 1) = x^{n+1} - x^n + x^3 - x^2 - x + 1 \). This is possible only if \( n = 3 \) and \( m = 5 \).

**Remark.** It is an old (though difficult) result that the polynomial \( x^n \pm x^k \pm 1 \) is either irreducible or equals \( x^2 \pm x + 1 \) times an irreducible factor.

7. To avoid working with cases, we use oriented angles modulo 180°. Let \( K \) be the circumcenter of \( \triangle BCD \), and \( X \) any point on the common tangent to the circles at \( D \). Since the tangents at the ends of a chord are equally inclined to the chord, we have \( \angle BAC = \angle ABD + \angle BDC + \angle DCA = \angle BDX + \angle BDC + \angle XDC = 2\angle BDC = \angle BKC \). It follows that \( B, C, A, K \) are concyclic, as required.

8. Construct equilateral triangles \( ACP \) and \( ABQ \) outside the triangle \( ABC \).

Since \( \angle APC + \angle AFC = 60° + 120° = 180° \), the points \( A, C, F, P \) lie on a circle; hence \( \angle AFP = \angle ACP = 60° = \angle AFD \), so \( D \) lies on the segment \( FP \); similarly, \( E \) lies on \( FQ \). Further, note that

\[
\frac{FP}{FD} = 1 + \frac{DP}{FD} = 1 + \frac{S_{APC}}{S_{AFC}} \geq 4
\]

with equality if \( F \) is the midpoint of the smaller arc \( AC \); hence \( FD \leq \frac{1}{4}FP \) and \( FE \leq \frac{1}{4}FQ \). Then by the law of cosines,

\[
DE = \sqrt{FD^2 + FE^2 + FD \cdot FE} 
\leq \frac{1}{4}\sqrt{FP^2 + FQ^2 + FP \cdot FQ} = \frac{1}{4}PQ \leq AP + AQ = AB + AC.
\]

Equality holds if and only if \( \triangle ABC \) is equilateral.

9. Since \( \angle BCA = \frac{1}{2}\angle BOA = \angle BOD \), the lines \( CA \) and \( OD \) are parallel, so that \( ODAI \) is a parallelogram. It follows that \( AI = OD = OE = AE = AF \). Hence
\[ \angle IFE = \angle IFA - \angle EFA = \angle AIF - \angle ECA = \angle AIF - \angle ACF = \angle CFI. \]

Also, from \( AE = AF \) we get that \( CI \) bisects \( \angle ECF \). Therefore \( I \) is the incenter of \( \triangle CEF \).

10. Let \( O \) be the circumcenter of \( A_1A_2C \), and \( O_1, O_2 \) the centers of \( S_1, S_2 \) respectively.

First, from \( \angle A_1QA_2 = 180^\circ - \angle PA_1Q - \angle QA_2P = \frac{1}{2}(360^\circ - \angle PO_1Q - \angle QO_2P) = \angle O_1QO_2 \) we obtain \( \angle A_1QA_2 = \angle B_1QB_2 = \angle O_1QO_2 \).

Therefore \( \angle A_1QA_2 = \angle B_1QP + \angle PQB_2 = \angle CA_1P + \angle CA_2P = 180^\circ - \angle A_1CA_2 \), from which we conclude that \( Q \) lies on the circumcircle of \( \triangle A_1A_2C \). Hence \( OA_1 = OQ \). However, we also have \( O_1A_2 = O_1Q \). Consequently, \( O, O_1 \) both lie on the perpendicular bisector of \( A_1Q \), so \( OO_1 \perp A_1Q \). Similarly, \( OO_2 \perp A_2Q \), leading to \( \angle O_2OO_1 = 180^\circ - \angle A_1QA_2 = 180^\circ - \angle O_1QO_2 \). Hence, \( O \) lies on the circle through \( O_1, O_2, Q \), which is fixed.

11. When \( S \) is the set of vertices of a regular pentagon, then it is easily verified that \( \frac{M(S)}{m(S)} = \frac{1 + \sqrt{5}}{2} = \alpha \). We claim that this is the best possible.

Let \( A, B, C, D, E \) be five arbitrary points, and assume that \( \triangle ABC \) has the area \( M(S) \). We claim that some triangle has area less than or equal to \( M(S)/\alpha \).

Construct a larger triangle \( A'B'C' \) with \( C \in A'B' \parallel AB \), \( A \in B'C' \parallel BC \), \( B \in C'A' \parallel CA \). The point \( D \), as well as \( E \), must lie on the same side of \( B'C' \) as \( BC \), for otherwise \( \triangle DBC \) would have greater area than \( \triangle ABC \).

A similar result holds for the other edges, and therefore \( D, E \) lie inside the triangle \( A'B'C' \) or on its boundary. Moreover, at least one of the triangles \( A'B'C, AB'C, ABC' \), say \( ABC' \), contains neither \( D \) nor \( E \). Hence we can assume that \( D, E \) are contained inside the quadrilateral \( A'B'AB \).

An affine linear transformation does not change the ratios between areas. Thus if we apply such an affine transformation mapping \( A, B, C \) into the vertices \( ABMCN \) of a regular pentagon, we won’t change \( M(S)/m(S) \). If now \( D \) or \( E \) lies inside \( ABMCN \), then we are done. Suppose that both \( D \) and \( E \) are inside the triangles \( CAM, CBN \). Then \( CD, CE \leq CM \) (because \( CM = CN = CA' = CB' \)) and \( \angle DCE \) is either less than or equal to \( 36^\circ \) or greater than or equal to \( 108^\circ \), from which we obtain that the area of \( \triangle CDE \) cannot exceed the area of \( \triangle CMN = M(S)/\alpha \). This completes the proof.

12. Let \( l(MN) \) denote the length of the shorter arc \( MN \) of a given circle.
Lemma. Let $PR, QS$ be two chords of a circle $k$ of radius $r$ that meet each other at a point $X$, and let $\angle PXQ = \angle RXS = 2\alpha$. Then $l(PQ) + l(RS) = 4\alpha r$.

Proof. Let $O$ be the center of the circle. Then $l(PQ) + l(RS) = \angle POQ \cdot r + \angle ROS \cdot r = 2(\angle QSP + \angle RPS)r = 2\angle QXP \cdot r = 4\alpha r$. Consider a circle $k$, sufficiently large, whose interior contains all the given circles. For any two circles $C_i, C_j$, let their exterior common tangents $PR, QS$ $(P, Q, R, S \in k)$ form an angle $2\alpha$. Then $O_iO_j = \frac{2}{\sin \alpha}$, so $\alpha > \sin \alpha = 2O_iO_j$. By the lemma we have $l(PQ) + l(RS) = 4\alpha r \geq \frac{8r}{O_iO_j}$, and hence

$$\frac{1}{O_iO_j} \leq \frac{l(PQ) + l(RS)}{8r}. \quad (1)$$

Now sum all these inequalities for $i < j$. The result will follow if we show that every point of the circle $k$ belongs to at most $n - 1$ arcs such as $PQ, RS$. Indeed, that will imply that the sum of all the arcs is at most $2(n - 1)\pi r$, hence from (1) we conclude that $\sum \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}$.

Consider an arbitrary point $T$ of $k$. We prove by induction (the basis $n = 1$ is trivial) that the number of pairs of circles that are simultaneously intercepted by a ray from $T$ is at most $n - 1$. Let $Tu$ be a ray touching $k$ at $T$. If we let this ray rotate around $T$, it will at some moment intercept a pair of circles for the first time, say $C_1, C_2$. At some further moment the interception with one of these circles, say $C_1$, is lost and never established again. Thus the pair $(C_1, C_2)$ is the only pair containing $C_1$ that is intercepted by some ray from $T$. On the other hand, by the inductive hypothesis the number of such pairs not containing $C_1$ does not exceed $n - 2$, justifying our claim.

13. Let $k$ be the circle through $B, C$ that is tangent to the circle $\Omega$ at point $N'$. We must prove that $K, M, N'$ are collinear. Since the statement is trivial for $AB = AC$, we may assume that $AC > AB$. As usual, $R, r, \alpha, \beta, \gamma$ denote the circumradius and the inradius and the angles of $\triangle ABC$, respectively.

We have $\tan \angle BKM = \frac{DM}{DK}$. Straightforward calculation gives $DM = \frac{1}{2}AD = R\sin \beta \sin \gamma$ and $DK = \frac{DC - DB}{2} - \frac{KC - KB}{2} = R\sin(\beta - \gamma) - R(\sin \beta - \sin \gamma) = 4R\sin \frac{\beta - \gamma}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$, so we obtain

$$\tan \angle BKM = \frac{\sin \beta \sin \gamma}{4 \sin \frac{\beta - \gamma}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}} = \frac{\cos \frac{\beta}{2} \cos \frac{\gamma}{2}}{\sin \frac{\beta - \gamma}{2}}.$$ 

To calculate the angle $BKN'$, we apply the inversion $\psi$ with center at $K$ and power $BK \cdot CK$. For each object $X$, we denote by $\hat{X}$ its image under $\psi$. The incircle $\Omega$ maps to a...
line $\hat{\Omega}$ parallel to $\hat{B}\hat{C}$, at distance $\frac{BK \cdot CK}{2r}$ from $\hat{B}\hat{C}$. Thus the point $\hat{N}'$ is the projection of the midpoint $\hat{U}$ of $\hat{B}\hat{C}$ onto $\hat{\Omega}$. Hence

$$\tan \angle BKN' = \tan \angle \hat{B}K\hat{N}' = \frac{\hat{U}\hat{N}'}{\hat{U}K} = \frac{BK \cdot CK}{r(CK - BK)}.$$ 

Again, one easily checks that $KB \cdot KC = bc\sin^2 \frac{\alpha}{2}$ and $r = 4R \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \sin \frac{\gamma}{2}$, which implies

$$\tan \angle BKN' = \frac{bc\sin^2 \frac{\alpha}{2}}{r(b - c)} = \frac{4R^2 \sin \beta \sin \gamma \sin^2 \frac{\alpha}{2}}{4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \cdot 2R(\sin \beta - \sin \gamma)} = \cos \frac{\beta}{2} \cos \frac{\gamma}{2}.$$ 

Hence $\angle BKM = \angle BKN'$, which implies that $K, M, N'$ are indeed collinear; thus $N' \equiv N$.

14. Let $G$ be the other point of intersection of the line $FK$ with the arc $BAD$. Since $BN/NC = DK/KB$ and $\angle CEB = \angle BGD$ the triangles $CEB$ and $BDG$ are similar. Thus $BN/NE = DK/KG = FK/KB$. From $BN = MK$ and $BK = MN$ it follows that $MN/NE = FK/KM$. But we also have that $\angle MNE = 90^\circ + \angle MNB = 90^\circ + \angle MKB = \angle FKM$, and hence $\triangle MNE \sim \triangle FKM$.

Now $\angle EMF = \angle NMK - \angle NME - \angle KMF = \angle NMK - \angle NME - \angle NEM = \angle NMK - 90^\circ + \angle BNM = 90^\circ$ as claimed.

15. We observe first that $f$ is surjective. Indeed, setting $y = -f(x)$ gives $f(f(-f(x)) - x) = f(0) - 2x$, where the right-hand expression can take any real value.

In particular, there exists $x_0$ for which $f(x_0) = 0$. Now setting $x = x_0$ in the functional equation yields $f(y) = 2x_0 + f(f(y) - x_0)$, so we obtain

$$f(z) = z - x_0 \quad \text{for} \quad z = f(y) - x_0.$$ 

Since $f$ is surjective, $z$ takes all real values. Hence for all $z$, $f(z) = z + c$ for some constant $c$, and this is indeed a solution.

16. For $n \geq 2$, let $(k_1, k_2, \ldots, k_n)$ be the permutation of $\{1, 2, \ldots, n\}$ with $a_{k_1} \leq a_{k_2} \leq \cdots \leq a_{k_n}$. Then from the condition of the problem, using the Cauchy–Schwarz inequality, we obtain
\[ c \geq a_{k_n} - a_{k_1} = |a_{k_n} - a_{k_{n-1}}| + \cdots + |a_{k_3} - a_{k_2}| + |a_{k_2} - a_{k_1}| \]
\[ \geq \frac{1}{k_1 + k_2} + \frac{1}{k_2 + k_3} + \cdots + \frac{1}{k_{n-1} + k_n} \]
\[ \geq \frac{(n-1)^2}{(k_1 + k_2) + (k_2 + k_3) + \cdots + (k_{n-1} + k_n)} \]
\[ = \frac{(n-1)^2}{2(k_1 + k_2 + \cdots + k_n) - k_1 - k_n} \geq \frac{(n-1)^2}{n^2 + n - 3} \geq \frac{n-1}{n+2}. \]

Therefore \( c \geq 1 - \frac{3}{n+2} \) for every positive integer \( n \). But if \( c < 1 \), this inequality is obviously false for all \( n > \frac{3}{1-c} - 2 \). We conclude that \( c \geq 1 \).

**Remark.** The least value of \( c \) is not greater than \( 2 \ln 2 \). An example of a sequence \( \{a_n\} \) with \( 0 \leq a_n \leq 2 \ln 2 \) can be constructed inductively as follows: Given \( a_1, a_2, \ldots, a_{n-1} \), then \( a_n \) can be any number from \([0, 2 \ln 2]\) that does not belong to any of the intervals \((a_i - \frac{1}{i+n}, a_i + \frac{1}{i+n})\) \( (i = 1, 2, \ldots, n - 1) \), and the total length of these intervals is always less than or equal to
\[ \frac{2}{n+1} + \frac{2}{n+2} + \cdots + \frac{2}{2n-1} < 2 \ln 2. \]

17. Let \( x, y \) be distinct integers satisfying \( xP(x) = yP(y) \); this is equivalent to \( a(x^4 - y^4) + b(x^3 - y^3) + c(x^2 - y^2) + d(x - y) = 0 \). Dividing by \( x - y \) we obtain
\[ a(x^3 + x^2y + xy^2 + y^3) + b(x^2 + xy + y^2) + c(x + y) + d = 0. \]
Putting \( x + y = p, x^2 + y^2 = q \) leads to \( x^2 + xy + y^2 = \frac{p^2 + q}{2} \), so the above equality becomes
\[ apq + \frac{b}{2}(p^2 + q) + cp + d = 0, \quad \text{i.e.} \quad (2ap + b)q = -(bp^2 + 2cp + 2d). \]

Since \( q \geq p^2/2 \), it follows that \( p^2|2ap + b| \leq 2|bp^2 + 2cp + 2d| \), which is possible only for finitely many values of \( p \), although there are infinitely many pairs \((x, y)\) with \( xP(x) = yP(y) \). Hence there exists \( p \) such that \( xP(x) = (p-x)P(p-x) \) for infinitely many \( x \), and therefore for all \( x \).

If \( p \neq 0 \), then \( p \) is a root of \( P(x) \). If \( p = 0 \), the above relation gives \( P(x) = -P(-x) \). This forces \( b = d = 0 \), so \( P(x) = x(ax^2 + c) \). Thus \( 0 \) is a root of \( P(x) \).

18. Putting \( x = z = 0 \) and \( t = y \) into the given equation gives \( 4f(0)f(y) = 2f(0) \) for all \( y \). If \( f(0) \neq 0 \), then we deduce \( f(y) = \frac{1}{2}, \) i.e., \( f \) is identically equal to \( \frac{1}{2} \).

Now we suppose that \( f(0) = 0 \). Setting \( z = t = 0 \) we obtain
\[ f(xy) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}. \quad (1) \]
Thus if \( f(y) = 0 \) for some \( y \neq 0 \), then \( f \) is identically zero. So, assume \( f(y) \neq 0 \) whenever \( y \neq 0 \).
Next, we observe that \( f \) is strictly increasing on the set of positive reals. Actually, it follows from (1) that \( f(x) = f(\sqrt{x})^2 \geq 0 \) for all \( x \geq 0 \), so that the given equation for \( t = x \) and \( z = y \) yields \( f(x^2+y^2) = (f(x)+f(y))^2 \geq f(x^2) \) for all \( x, y \geq 0 \).

Using (1) it is easy to get \( f(1) = 1 \). Now plugging \( t = y \) into the given equation, we are led to

\[
2[f(x) + f(z)] = f(x - z) + f(x + z) \quad \text{for all } x, z.
\]

(2)

In particular, \( f(z) = f(-z) \). Further, it is easy to get by induction from (2) that \( f(nx) = n^2f(x) \) for all integers \( n \) (and consequently for all rational numbers as well). Therefore \( f(q) = f(-q) = q^2 \) for all \( q \in \mathbb{Q} \). But \( f \) is increasing for \( x > 0 \), so we must have \( f(x) = x^2 \) for all \( x \).

It is easy to verify that \( f(x) = 0, f(x) = \frac{1}{2} \) and \( f(x) = x^2 \) are indeed solutions.

19. Write \( m = [\sqrt[n]{n}] \). To simplify the calculation, we shall assume that \([b] = 1\). Then \( a = \sqrt[n]{n}, \ b = \frac{1}{\sqrt[n]{n-m}} = \frac{1}{n-m} (m^2 + m\sqrt[n]{n} + \sqrt[n]{n^2}) \), \( c = \frac{1}{n-m} = u + v\sqrt[n]{n} + w\sqrt[n]{n^2} \) for certain rational numbers \( u, v, w \). Obviously, integers \( r, s, t \) with \( ra + sb + tc = 0 \) exist if (and only if) \( u = m^2w, i.e., if (b - 1)(m^2w + v\sqrt[n]{n} + w\sqrt[n]{n^2}) = 1 \) for some rational \( v, w \).

When the last equality is expanded and simplified, comparing the coefficients at \( 1, \sqrt[n]{n}, \sqrt[n]{n^2} \) one obtains

\[
\sqrt[n]{n} : (m^2 + m^3 - n)v + (m^3 + n)w = 0,
\]

\[
\sqrt[n]{n^2} : mv + (2m^2 + m^3 - n)w = 0.
\]

In order for the system (1) to have a solution \( v, w \), we must have \((2m^2 + m^3 - n)(m^2 + m^3 - n) = m(m^3 + n)\). This quadratic equation has solutions \( n = m^3 \) and \( n = m^3 + 3m^2 + m \). The former is not possible, but the latter gives \( a - \{a\} > \frac{1}{2} \), so \([b] = 1\), and the system (1) in \( v, w \) is solvable. Hence every number \( n = m^3 + 3m^2 + m, m \in \mathbb{N} \), satisfies the condition of the problem.

20. Assume to the contrary that \( \frac{1}{a_1} + \cdots + \frac{1}{a_n} > 1 \). Certainly \( n \geq 2 \) and \( A \) is infinite. Define \( f_i : A \to A \) as \( f_i(x) = b_ix + c_i \) for each \( i \). By condition (ii), \( f_i(x) = f_j(y) \) implies \( i = j \) and \( x = y \); iterating this argument, we deduce that \( f_{i_1}(\cdots f_{i_m}(x)\cdots) = f_{j_1}(\cdots f_{j_m}(x)\cdots) \) implies \( i_1 = j_1, \ldots, i_m = j_m \) and \( x = y \).

As an illustration, we shall consider the case \( b_1 = b_2 = b_3 = 2 \) first. If \( a \) is large enough, then for any \( i_1, \ldots, i_m \in \{1, 2, 3\} \) we have \( f_{i_1} \circ \cdots \circ f_{i_m}(a) \leq 21^m a \). However, we obtain \( 3^m \) values in this way, so they cannot all be distinct if \( m \) is sufficiently large, a contradiction.

In the general case, let real numbers \( d_i > b_i, i = 1, 2, \ldots, n \), be chosen such that \( \frac{1}{d_1} + \cdots + \frac{1}{d_n} > 1 \): for \( a \) large enough, \( f_i(x) < d_i a \) for each \( x \geq a \).
Also, let $k_i > 0$ be arbitrary rational numbers with sum 1; denote by $N_0$ the least common multiple of their denominators.

Let $N$ be a fixed multiple of $N_0$, so that each $k_i N$ is an integer. Consider all combinations $f_i \circ \cdots \circ f_{iN}$ of $N$ functions, among which each $f_i$ appears exactly $k_i N$ times. There are $F_N = \frac{N!}{(k_1 N)! \cdots (k_n N)!}$ such combinations, so they give $F_N$ distinct values when applied to $a$. On the other hand, $f_i \circ \cdots \circ f_{iN}(a) \leq (d_{k_1}^{k_1} \cdots d_{kn}^{k_n})^N a$. Therefore

\[(d_{k_1}^{k_1} \cdots d_{kn}^{k_n})^N a \geq F_N \quad \text{for all } N, N_0 \mid N.\] (1)

It remains to find a lower estimate for $F_N$. In fact, it is straightforward to verify that $F_N + N_0 / F_N$ tends to $Q^{N_0}$, where $Q = 1 / (k_1 d_1 \cdots k_n d_n)$. Consequently, for every $q < Q$ there exists $p > 0$ such that $F_N > pq^N$. Then (1) implies that

\[\left(\frac{d_{k_1}^{k_1} \cdots d_{kn}^{k_n}}{q}\right)^N a > \frac{p}{a} \quad \text{for every multiple } N \text{ of } N_0,
\]

and hence $d_{k_1}^{k_1} \cdots d_{kn}^{k_n} / q \geq 1$. This must hold for every $q < Q$, and so we have $d_{k_1}^{k_1} \cdots d_{kn}^{k_n} \geq Q$, i.e.,

\[(k_1 d_1)^{k_1} \cdots (k_n d_n)^{k_n} \geq 1.
\]

However, if we choose $k_1, \ldots, k_n$ such that $k_1 d_1 = \cdots = k_n d_n = u$, then we must have $u \geq 1$. Therefore $\frac{1}{d_1} + \cdots + \frac{1}{d_n} \leq k_1 + \cdots + k_n = 1$, a contradiction.

21. Let $a_i$ be the number of blue points with $x$-coordinate $i$, and $b_i$ the number of blue points with $y$-coordinate $i$. Our task is to show that $a_0 a_1 \cdots a_{n-1} = b_0 b_1 \cdots b_{n-1}$. Moreover, we claim that $a_0, \ldots, a_{n-1}$ is a permutation of $b_0, \ldots, b_{n-1}$, and to show this we use induction on the number of red points.

The result is trivial if all the points are blue. So, choose a red point $(x, y)$ with $x + y$ maximal: clearly $a_x = b_y = n - x - y - 1$. If we change this point to blue, $a_x$ and $b_y$ will decrease by 1. Then by the induction hypothesis, $a_0, \ldots, a_{n-1}$ with $a_x$ decreased by 1 is a permutation of $b_0, \ldots, b_{n-1}$ with $b_y$ decreased by 1. However, $a_x = b_y$, and the claim follows.

Remark. One can also use induction on $n$: it is not more difficult.

22. Write $n = 2k + 1$. Consider the black squares at an odd height: there are $(k + 1)^2$ of them in total and no two can be covered by one trimino. Thus, we always need at least $(k + 1)^2$ triminoes, which cover $3(k + 1)^2$ squares in total. However, $3(k + 1)^2$ is greater than $n^2$ for $n = 1, 3, 5$, so we must have $n \geq 7$.

The case $n = 7$ admits such a covering, as shown in Figure 1. For $n > 7$ this is possible as well: it follows by induction from Figure 2.
23. We claim that there are \( n! \) full sequences. To show this, we construct a bijection with the set of permutations of \( \{1, 2, \ldots, n\} \).

Consider a full sequence \((a_1, a_2, \ldots, a_n)\), and let \( m \) be the greatest of the numbers \( a_1, \ldots, a_n \). Let \( S_k, 1 \leq k \leq m \), be the set of those indices \( i \) for which \( a_i = k \). Then \( S_1, \ldots, S_m \) are nonempty and form a partition of the set \( \{1, 2, \ldots, n\} \). Now we write down the elements of \( S_1 \) in descending order, then the elements of \( S_2 \) in descending order and so on. This maps the full sequence to a permutation of \( \{1, 2, \ldots, n\} \). Moreover, this map is reversible, since each permutation uniquely breaks apart into decreasing sequences \( S'_1, S'_2, \ldots, S'_m \), so that \( \max S'_i > \min S'_{i-1} \). Therefore the full sequences are in bijection with the permutations of \( \{1, 2, \ldots, n\} \).

Second solution. Let there be given a full sequence of length \( n \). Removing from it the first occurrence of the highest number, we obtain a full sequence of length \( n - 1 \). On the other hand, each full sequence of length \( n - 1 \) can be obtained from exactly \( n \) full sequences of length \( n \). Therefore, if \( x_n \) is the number of full sequences of length \( n \), we deduce \( x_n = nx_{n-1} \).

24. Two moves are not sufficient. Indeed, the answer to each move is an even number between 0 and 54, so the answer takes at most 28 distinct values. Consequently, two moves give at most \( 28^2 = 784 \) distinct outcomes, which is less than \( 10^3 = 1000 \).

We now show that three moves are sufficient. With the first move \((0, 0, 0)\), we get the reply \( 2(x + y + z) \), so we now know the value of \( s = x + y + z \). Now there are several cases:

(i) \( s \leq 9 \). Then we ask \((9, 0, 0)\) as the second move and get \( (9 - x - y) + (9 - x - z) + (y + z) = 18 - 2x \), so we come to know \( x \). Asking \((0, 9, 0)\) we obtain \( y \), which is enough, since \( z = s - x - y \).

(ii) \( 10 \leq s \leq 17 \). In this case the second move is \((9, s - 9, 0)\). The answer is \( z + (9 - x) + |x + z - 9| = 2k \), where \( k = z \) if \( x + z \geq 9 \), or \( k = 9 - x \) if \( x + z < 9 \). In both cases we have \( z \leq k \leq y + z \leq s \).

Let \( s - k \leq 9 \). Then in the third move we ask \((s - k, 0, k)\) and obtain \( |z - k| + |k - y - z| + y \), which is actually \((k - z) + (y + z - k) + y = 2y \).

Thus we also find out \( x + z \), and thus deduce whether \( k \) is \( z \) or \( 9 - x \). Consequently we determine both \( x \) and \( z \).

Let \( s - k > 9 \). In this case, the third move is \((9, s - k - 9, k)\). The answer is \(|s - k - x - y| + |s - 9 - y - z| + |k + 9 - z - x| = (k - z) + (9 - x) + (9 - x + k - z) = 18 + 2k - 2(x + z)\), from which we find out again whether \( k \) is \( z \) or \( 9 - x \). Now we are easily done.
(iii) $18 \leq s \leq 27$. Then as in the first case, asking $(0, 9, 9)$ and $(9, 0, 9)$ we obtain $x$ and $y$.

25. Assume to the contrary that no set of size less than $r$ meets all sets in $\mathcal{F}$. Consider any set $A$ of size less than $r$ that is contained in infinitely many sets of $\mathcal{F}$. By the assumption, $A$ is disjoint from some set $B \in \mathcal{F}$. Then of the infinitely many sets that contain $A$, each must meet $B$, so some element $b$ of $B$ belongs to infinitely many of them. But then the set $A \cup \{b\}$ is contained in infinitely many sets of $\mathcal{F}$ as well.

Such a set $A$ exists: for example, the empty set. Now taking for $A$ the largest such set we come to a contradiction.

26. Write $n = 2m$. We shall define a directed graph $G$ with vertices $1, \ldots, m$ and edges labelled $1, 2, \ldots, 2m$ in such a way that the edges issuing from $i$ are labelled $2i - 1$ and $2i$, and those entering it are labelled $i$ and $i + m$. What we need is an Euler circuit in $G$, namely a closed path that passes each edge exactly once. Indeed, if $x_i$ is the $i$th edge in such a circuit, then $x_i$ enters some vertex $j$ and $x_{i+1}$ leaves it, so $x_i \equiv j \pmod{m}$ and $x_{i+1} = 2j - 1$ or $2j$. Hence $2x_i \equiv 2j$ and $x_{i+1} \equiv 2x_i$ or $2x_i - 1 \pmod{n}$, as required.

The graph $G$ is connected: by induction on $k$ there is a path from $1$ to $k$, since $1$ is connected to $j$ with $2j = k$ or $2j - 1 = k$, and there is an edge from $j$ to $k$. Also, the in-degree and out-degree of each vertex of $G$ are equal (to 2), and thus by a known result, $G$ contains an Euler circuit.

27. For a graph $G$ on 120 vertices (i.e., people at the party), write $q(G)$ for the number of weak quartets in $G$. Our solution will consist of three parts. First, we prove that some graph $G$ with maximal $q(G)$ breaks up as a disjoint union of complete graphs. This will follow if we show that any two adjacent vertices $x, y$ have the same neighbors (apart from themselves). Let $G_x$ be the graph obtained from $G$ by “copying” $x$ to $y$ (i.e., for each $z \neq x, y$, we add the edge $zy$ if $zx$ is an edge, and delete $zy$ if $zx$ is not an edge). Similarly $G_y$ is the graph obtained from $G$ by copying $y$ to $x$. We claim that $2q(G) \leq q(G_x) + q(G_y)$. Indeed, the number of weak quartets containing neither $x$ nor $y$ is the same in $G, G_x$, and $G_y$, while the number of those containing both $x$ and $y$ is not less in $G_x$ and $G_y$ than in $G$. Also, the number containing exactly one of $x$ and $y$ in $G_x$ is at least twice the number in $G$ containing $x$ but not $y$, while the number containing exactly one of $x$ and $y$ in $G_y$ is at least twice the number in $G$ containing $y$ but not $x$. This justifies our claim by adding. It follows that for an extremal graph $G$ we must have $q(G) = q(G_x) = q(G_y)$. Repeating the copying operation pair by pair ($y$ to $x$, then their common neighbor $z$ to both $x, y$, etc.) we eventually obtain an extremal graph consisting of disjoint complete graphs.

Second, suppose the complete graphs in $G$ have sizes $a_1, a_2, \ldots, a_n$. Then
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\[
q(G) = \sum_{i=1}^{n} \left( \frac{a_i}{2} \right) \sum_{\substack{j<k \\
j,k \neq i}} a_j a_k.
\]

If we fix all the \(a_i\) except two, say \(p, q\), then \(p + q = s\) is fixed, and for some constants \(C_i\),

\[
q(G) = C_1 + C_2 pq + C_3 \left( \binom{p}{2} + \binom{q}{2} \right) + C_4 \left( \binom{p}{2} + p \binom{q}{2} \right) = A + Bpq,
\]

where \(A\) and \(B\) depend only on \(s\). Hence the maximum of \(q(G)\) is attained if \(|p - q| \leq 1\) or \(pq = 0\). Thus if \(q(G)\) is maximal, any two nonzero \(a_i\)'s differ by at most 1.

Finally, if \(G\) consists of \(n\) disjoint complete graphs, then \(q(G)\) cannot exceed the value obtained if \(a_1 = \cdots = a_n\) (not necessarily integral), which equals

\[
Q_n = \frac{120^2}{n} \left( \frac{120/n}{2} \right) \left( \frac{n - 1}{2} \right) = 30 \cdot \frac{120^2(n - 1)(n - 2)(120 - n)}{n^3}.
\]

It is easy to check that \(Q_n\) takes its maximum when \(n = 5\) and \(a_1 = \cdots = a_5 = 24\), and that this maximum equals \(15 \cdot 23 \cdot 24^3 = 4769280\).
4.44 Solutions to the Shortlisted Problems of IMO 2003

1. Consider the points \( O(0,0,0) \), \( P(a_{11}, a_{21}, a_{31}) \), \( Q(a_{12}, a_{22}, a_{32}) \), \( R(a_{13}, a_{23}, a_{33}) \) in three-dimensional Euclidean space. It is enough to find a point \( U(u_1, u_2, u_3) \) in the interior of the triangle \( PQR \) whose coordinates are all positive, all negative, or all zero (indeed, then we have \( \overrightarrow{OU} = c_1 \overrightarrow{OP} + c_2 \overrightarrow{OQ} + c_3 \overrightarrow{OR} \) for some \( c_1, c_2, c_3 > 0 \) with \( c_1 + c_2 + c_3 = 1 \).

Let \( P'(a_{11}, a_{21}, 0) \), \( Q'(a_{12}, a_{22}, 0) \), and \( R'(a_{13}, a_{23}, 0) \) be the projections of \( P, Q, \) and \( R \) onto the \( Oxy \) plane. We see that \( P', Q', R' \) lie in the fourth, second, and third quadrants, respectively. We have the following two cases:

(i) \( O \) is in the exterior of \( \triangle P'Q'R' \).

Set \( S' = OR' \cap P'Q' \) and let \( S \) be the point of the segment \( PQ \) that projects to \( S' \). The point \( S \) has its \( z \) coordinate negative (because the \( z \) coordinates of \( P \) and \( Q \) are negative). Thus any point of the segment \( SR \) sufficiently close to \( S \) has all coordinates negative.

(ii) \( O \) is in the interior or on the boundary of \( \triangle P'Q'R' \).

Let \( T \) be the point in the plane \( PQR \) whose projection is \( O \). If \( T = O \), then all coordinates of \( T \) are zero, and we are done. Otherwise \( O \) is interior to \( \triangle P'Q'R' \). Suppose that the \( z \) coordinate of \( T \) is positive (negative). Since \( x \) and \( y \) coordinates of \( T \) are equal to 0, there is a point \( U \) inside \( PQR \) close to \( T \) with both \( x \) and \( y \) coordinates positive (respectively negative), and this point \( U \) has all coordinates of the same sign.

2. We can rewrite (ii) as \(-f(a-1)(f(b)-1) = f(-(a-1)(b-1)) + 1 - 1 \).

So putting \( g(x) = f(x+1) - 1 \), this equation becomes \(-g(a-1)g(b-1) = g(-(a-1)(b-1)) \) for \( a < 1 < b \). Hence

\[ g(x)g(y) = g(-xy) \] for \( x < 0 < y \),

and \( g \) is nondecreasing with \( g(-1) = -1, g(0) = 0 \). \hspace{1cm} (1)

Conversely, if \( g \) satisfies (1), then \( f \) is a solution of our problem.

Setting \( y = 1 \) in (1) gives \(-g(-x)g(1) = g(x) \) for each \( x > 0 \), and therefore (1) reduces to \( g(1)g(yz) = g(y)g(z) \) for all \( y, z > 0 \). We have two cases:

(i) \( g(1) = 0 \). By (1) we have \( g(z) = 0 \) for all \( z > 0 \). Then any nondecreasing function \( g : \mathbb{R} \to \mathbb{R} \) with \( g(-1) = -1 \) and \( g(z) = 0 \) for \( z \geq 0 \) satisfies (1) and gives a solution: \( f \) is nondecreasing, \( f(0) = 0 \) and \( f(x) = 1 \) for every \( x \geq 1 \)

(ii) \( g(1) \neq 0 \). Then the function \( h(x) = \frac{g(x)}{g(1)} \) is nondecreasing and satisfies \( h(0) = 0, h(1) = 1, \) and \( h(xy) = h(x)h(y) \). Fix \( a > 0 \), and let \( h(a) = b = a^k \) for some \( k \in \mathbb{R} \). It follows by induction that \( h(a^q) = h(a)^q = \ldots \)
(a^q)^k \) for every rational number \( q \). But \( h \) is nondecreasing, so \( k \geq 0 \), and since the set \( \{a^q \mid q \in \mathbb{Q}\} \) is dense in \( \mathbb{R}^+ \), we conclude that \( h(x) = x^k \) for every \( x > 0 \). Finally, putting \( g(1) = c \), we obtain \( g(x) = cx^k \) for all \( x > 0 \). Then \( g(-x) = -x^k \) for all \( x > 0 \). This \( g \) obviously satisfies (1). Hence

\[
f(x) = \begin{cases} 
  c(x-1)^k, & \text{if } x > 1; \\
  1, & \text{if } x = 1; \\
  1 - (1-x)^k, & \text{if } x < 1,
\end{cases}
\]

where \( c > 0 \) and \( k \geq 0 \).

3. (a) Given any sequence \( c_n \) (in particular, such that \( C_n \) converges), we shall construct \( a_n \) and \( b_n \) such that \( A_n \) and \( B_n \) diverge.

First, choose \( n_1 \) such that \( n_1c_1 > 1 \) and set \( a_1 = a_2 = \cdots = a_{n_1} = c_1 \); this uniquely determines \( b_2 = c_2, \ldots, b_{n_1} = c_{n_1} \). Next, choose \( n_2 \) such that \( (n_2 - n_1)c_{n_1+1} > 1 \) and set \( b_{n_1+1} = \cdots = b_{n_2} = c_{n_1+1} \); again \( a_{n_1+1}, \ldots, a_{n_2} \) is hereby determined. Then choose \( n_3 \) with \( (n_3 - n_2)c_{n_2+1} > 1 \) and set \( a_{n_2+1} = \cdots = a_{n_3} = c_{n_2+1} \), and so on. It is plain that in this way we construct decreasing sequences \( a_n, b_n \) such that \( \sum a_n \) and \( \sum b_n \) diverge, since they contain an infinity of subsums that exceed 1; on the other hand, \( c_n = \min(a_n, b_n) \) and \( C_n \) is convergent.

(b) The answer changes in this situation. Suppose to the contrary that there is such a pair of sequences \( (a_n) \) and \( (b_n) \). There are infinitely many indices \( i \) such that \( c_i = b_i \) (otherwise all but finitely many terms of the sequence \( (c_n) \) would be equal to the terms of the sequence \( (a_n) \), which has an unbounded sum). Thus for any \( n_0 \in \mathbb{N} \) there is \( j \geq 2n_0 \) such that \( c_j = b_j \). Then we have

\[
\sum_{k=n_0}^{j} c_k \geq \sum_{k=n_0}^{j} c_j = (j - n_0) \frac{1}{j} \geq \frac{1}{2}.
\]

Hence the sequence \( (C_n) \) is unbounded, a contradiction.

4. By the Cauchy–Schwarz inequality we have

\[
\left( \sum_{i,j=1}^{n} (i-j)^2 \right)^{\frac{1}{2}} \left( \sum_{i,j=1}^{n} (x_i - x_j)^2 \right)^{\frac{1}{2}} \geq \left( \sum_{i,j=1}^{n} |i-j| \cdot |x_i - x_j| \right)^{\frac{1}{2}}.
\]

On the other hand, it is easy to prove (for example by induction) that

\[
\sum_{i,j=1}^{n} (i-j)^2 = (2n-2) \cdot 1^2 + (2n-4) \cdot 2^2 + \cdots + 2 \cdot (n-1)^2 \frac{n^2(n^2-1)}{6}
\]

and that

\[
\sum_{i,j=1}^{n} |i-j| \cdot |x_i - x_j| = \frac{n}{2} \sum_{i,j=1}^{n} |x_i - x_j|.
\]
Thus the inequality (1) becomes
\[
\frac{n^2(n^2 - 1)}{6} \left( \sum_{i,j=1}^{n} (x_i - x_j)^2 \right) \geq \frac{n^2}{4} \left( \sum_{i,j=1}^{n} |x_i - x_j| \right)^2,
\]
which is equivalent to the required one.

5. Placing \( x = y = z = 1 \) in (i) leads to \( 4f(1) = f(1)^3 \), so by the condition \( f(1) > 0 \) we get \( f(1) = 2 \). Also putting \( x = ts, y = \frac{t}{s}, z = \frac{s}{t} \) in (i) gives
\[
f(t)f(s) = f(ts) + f(t/s).
\]

In particular, for \( s = 1 \) the last equality yields \( f(t) = f(1/t) \); hence \( f(t) \geq f(1) = 2 \) for each \( t \). It follows that there exists \( g(t) \geq 1 \) such that \( f(t) = g(t) + \frac{1}{g(t)} \). Now it follows by induction from (1) that \( g(t^n) = g(t)^n \) for every integer \( n \), and therefore \( g(t^q) = g(t)^q \) for every rational \( q \). Consequently, if \( t > 1 \) is fixed, we have \( f(t^q) = a^q + a^{-q} \), where \( a = g(t) \).

But since the set of \( a^q \) (\( q \in \mathbb{Q} \)) is dense in \( \mathbb{R}^+ \) and \( f \) is monotone on \( (0, 1] \) and \( [1, \infty) \), it follows that \( f(t^r) = a^r + a^{-r} \) for every real \( r \). Therefore, if \( k \) is such that \( t^k = a \), we have
\[
f(x) = x^k + x^{-k} \quad \text{for every } x \in \mathbb{R}.
\]

6. Set \( X = \max\{x_1, \ldots, x_n\} \) and \( Y = \max\{y_1, \ldots, y_n\} \). By replacing \( x_i \) by \( x_i' = \frac{x_i}{X} \), \( y_i \) by \( y_i' = \frac{y_i}{Y} \) and \( z_i \) by \( z_i' = \frac{z_i}{XY} \), we may assume that \( X = Y = 1 \). It is sufficient to prove that
\[
M + z_2 + \cdots + z_{2n} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n,
\]

because this implies the result by the A-G mean inequality.

To prove (1) it is enough to prove that for any \( r \), the number of terms greater than \( r \) on the left-hand side of (1) is at least that number on the right-hand side of (1).

If \( r \geq 1 \), then there are no terms on the right-hand side greater than \( r \). Suppose that \( r < 1 \) and consider the sets \( A = \{i \mid 1 \leq i \leq n, \ x_i > r\} \) and \( B = \{i \mid 1 \leq i \leq n, \ y_i > r\} \). Set \( a = |A| \) and \( b = |B| \). If \( x_i > r \) and \( y_j > r \), then \( z_{i+j} > \sqrt{x_i y_j} > r \); hence
\[
C = \{k \mid 2 \leq k \leq 2n, \ z_k > r\} \supseteq A + B = \{\alpha + \beta \mid \alpha \in A, \beta \in B\}.
\]

It is easy to verify that \( |A + B| \geq |A| + |B| - 1 \). It follows that the number of \( z_k \)'s greater than \( r \) is \( \geq a + b - 1 \). But in that case \( M > r \), implying that at least \( a + b \) elements of the left-hand side of (1) is greater than \( r \), which completes the proof.

7. Consider the set \( D = \{x - y \mid x, y \in A\} \). Obviously, the number of elements of the set \( D \) is less than or equal to \( 101 \cdot 100 + 1 \). The sets \( A + t_i \) and \( A + t_j \)
are disjoint if and only if $t_i - t_j \notin D$. Now we shall choose inductively 100 elements $t_1, \ldots, t_{100}$.

Let $t_1$ be any element of the set $S \setminus D$ (such an element exists, since the number of elements of $S$ is greater than the number of elements of $D$).

Suppose now that we have chosen $k$ ($k \leq 99$) elements $t_1, \ldots, t_k$ from $D$ such that the difference of any two of the chosen elements does not belong to $D$. We can select $t_{k+1}$ to be an element of $S$ that does not belong to any of the sets $t_1 + D, t_2 + D, \ldots, t_k + D$ (this is possible to do, since each of the previous sets has at most $101 \cdot 100 + 1$ elements; hence their union has at most $99(101 \cdot 100 + 1) = 999999 < 1000000$ elements).

8. Let $S$ be the disk with the smallest radius, say $s$, and $O$ the center of that disk. Divide the plane into 7 regions: one bounded by disk $s$ and 6 regions $T_1, \ldots, T_6$ shown in the figure.

Any of the disks different from $S$, say $D_k$, has its center in one of the seven regions. If its center is inside $S$ then $D_k$ contains point $O$. Hence the number of disks different from $S$ having their centers in $S$ is at most 2002.

Consider a disk $D_k$ that intersects $S$ and whose center is in the region $T_i$. Let $P_i$ be the point such that $OP_i$ bisects the region $T_i$ and $OP_i = s\sqrt{3}$.

We claim that $D_k$ contains $P_i$. Divide the region $T_i$ by a line $l_i$ through $P_i$ perpendicular to $OP_i$ into two regions $U_i$ and $V_i$, where $O$ and $U_i$ are on the same side of $l_i$. Let $K$ be the center of $D_k$. Consider two cases:

(i) $K \in U_i$. Since the disk with the center $P_i$ and radius $s$ contains $U_i$, we see that $KP_i \leq s$. Hence $D_k$ contains $P_i$.

(ii) $K \in V_i$. Denote by $L$ the intersection point of the segment $KO$ with the circle $s$.

We want to prove that $KL > KP_i$. It is enough to prove that $\angle KPL > \angle KLP_i$. However, it is obvious that $\angle LP_iO \leq 30^\circ$ and $\angle LOP_i \leq 30^\circ$, hence $\angle KLP_i \leq 60^\circ$, while $\angle NP_iL = 90^\circ - \angle LP_iO \geq 60^\circ$. This implies that $\angle KPL \geq \angle NP_iL \geq 60^\circ \geq \angle KLP_i$ ($N$ is the point on the edge of $T_i$ as shown in the figure). Our claim is thus proved.

Now we see that the number of disks with centers in $T_i$ that intersect $S$ is less than or equal to 2003, and the total number of disks that intersect $S$ is not greater than $2002 + 6 \cdot 2003 = 7 \cdot 2003 - 1$.

9. Suppose that $k$ of the angles of an $n$-gon are right. Since the other $n - k$ angles are less than $360^\circ$ and the sum of the angles is $(n - 2)180^\circ$, we have
the inequality \( k \cdot 90^\circ + (n - k)360^\circ > (n - 2)180^\circ \), which is equivalent to \( k < \frac{2n+4}{3} \). Since \( n \) and \( k \) are integers, it follows that \( k \leq \left\lfloor \frac{2n}{3} \right\rfloor + 1 \).

If \( n = 5 \), then \( \left\lfloor \frac{2n}{3} \right\rfloor + 1 = 4 \), but if a pentagon has four right angles, the other angle is equal to \( 180^\circ \), which is impossible. Hence for \( n = 5 \), \( k \leq 3 \). It is easy to construct a pentagon with 3 right angles, e.g., as in the picture below.

Now we shall show by induction that for \( n \geq 6 \) there is an \( n \)-gon with \( \left\lfloor \frac{2n}{3} \right\rfloor + 1 \) internal right angles. For \( n = 6, 7, 8 \) examples are presented in the picture. Assume that there is a \((n-3)\)-gon with \( \left\lfloor \frac{2(n-3)}{3} \right\rfloor + 1 = \left\lfloor \frac{2n}{3} \right\rfloor - 1 \) internal right angles. Then one of the internal angles, say \( \angle BAC \), is not convex. Interchange the vertex \( A \) with four new vertices \( A_1, A_2, A_3, A_4 \) as shown in the picture such that \( \angle B A_1 A_2 = \angle A_3 A_4 C = 90^\circ \).

10. Denote by \( b_{ij} \) the entries of the matrix \( B \). Suppose the contrary, i.e., that there is a pair \((i_0, j_0)\) such that \( a_{i_0,j_0} \neq b_{i_0,j_0} \). We may assume without loss of generality that \( a_{i_0,j_0} = 0 \) and \( b_{i_0,j_0} = 1 \).

Since the sums of elements in the \( i_0 \)-th rows of the matrices \( A \) and \( B \) are equal, there is some \( j_1 \) for which \( a_{i_0,j_1} = 1 \) and \( b_{i_0,j_1} = 0 \). Similarly, from the fact that the sums in the \( j_1 \)-th columns of the matrices \( A \) and \( B \) are equal, we conclude that there exists \( i_1 \) such that \( a_{i_1,j_1} = 0 \) and \( b_{i_1,j_1} = 1 \). Continuing this procedure, we construct two sequences \( i_k, j_k \) such that \( a_{i_k,j_k} = 0 \), \( b_{i_k,j_k} = 1 \), \( a_{i_k,j_k+1} = 1 \), \( b_{i_k,j_k+1} = 0 \). Since the set of the pairs \((i_k, j_k)\) is finite, there are two different numbers \( t, s \) such that \((i_t, j_t) = (i_s, j_s)\). From the given condition we have that \( x_{i_t} + y_{i_t} < 0 \) and \( x_{i_{t+1}} + y_{i_{t+1}} \geq 0 \). But \( j_t = j_s \), and hence \( 0 \leq \sum_{k=s}^{t-1} (x_{i_k} + y_{j_{k+1}}) = \sum_{k=s}^{t-1} (x_{i_k} + y_{j_k}) < 0 \), a contradiction.

11. (a) By the pigeonhole principle there are two different integers \( x_1, x_2, x_1 > x_2 \), such that \(|\{x_1 \sqrt{3}\} - \{x_2 \sqrt{3}\}| < 0.001\). Set \( a = x_1 - x_2 \). Consider the equilateral triangle with vertices \((0,0), (2a,0), (a,a \sqrt{3})\).

The points \((0,0)\) and \((2a,0)\) are lattice points, and we claim that the point \((a,a \sqrt{3})\) is at distance less than 0.001 from a lattice point. Indeed, since 0.001 > \(|\{x_1 \sqrt{3}\} - \{x_2 \sqrt{3}\}| = a \sqrt{3} - (\{x_1 \sqrt{3}\} - \{x_2 \sqrt{3}\})\), we see that the distance between the points \((a,a \sqrt{3})\) and \((a, \{x_1 \sqrt{3}\} - \{x_2 \sqrt{3}\})\) is less than 0.001, and the point \((a, \{x_1 \sqrt{3}\} - \{x_2 \sqrt{3}\})\) is with integer coefficients.

(b) Suppose that \( P'Q'R' \) is an equilateral triangle with side length \( l \leq 96 \) such that each of its vertices \( P', Q', R' \) lies in a disk of radius 0.001 centered at a lattice point. Denote by \( P, Q, R \) the centers of these disks. Then we have \( l - 0.002 \leq PQ, QR, RP \leq l + 0.002 \). Since \( PQR \) is not an equilateral triangle, two of its sides are different, say...
PQ \neq QR. On the other hand, PQ^2, QR^2 are integers, so we have
1 \leq |PQ^2 - QR^2| = (PQ + QR)|PQ - QR| \leq \frac{0.004}{(2l + 0.004) \cdot 0.004} \leq 2 \cdot 96.002 = 0.004 < 1, which is a contradiction.

12. Denote by \overline{a_{k-1}a_{k-2}...a_0} the decimal representation of a number whose
digits are \(a_{k-1}, \ldots, a_0\). We will use the following well-known fact:
\[
\overline{a_{k-1}a_{k-2}...a_0} \equiv i \pmod{11} \iff \sum_{l=0}^{k-1} (-1)^l a_l \equiv i \pmod{11}.
\]
Let \(m\) be a positive integer. Define \(A\) as the set of integers \(n \leq n < 10^{2m}\) whose
right \(2^m - 1\) digits can be so permuted to yield an integer
divisible by 11, and \(B\) as the set of integers \(n \leq n < 10^{2m} - 1\) whose
digits can be permuted resulting in an integer divisible by 11.
Suppose that \(a = \overline{a_{2m-1}...a_0} \in A\). Then there that satisfies
\[
\sum_{l=0}^{2m-1} (-1)^l b_l \equiv 0 \pmod{11}.
\]
The \(2m\)-tuple \((b_{2m-1}, \ldots, b_0)\) satisfies \((1)\) if and only if the \(2m\)-tuple
\((kb_{2m-1} + l, \ldots, kb_0 + l)\) satisfies \((1)\), where \(k, l \in \mathbb{Z}, 1 \nmid k\).
Since \(a_0 + 1 \neq 0 \pmod{11}\), we can choose \(k\) from the set \(\{1, \ldots, 10\}\) such that
\(a_0 k + 1 \equiv 1 \pmod{11}\). Thus there is a permutation of the \(2m\)-tuple
\((a_{2m-1} + 1, \ldots, (a_1 + 1)k - 1, 0)\) satisfying \((1)\). Interchanging odd and
even positions if necessary, we may assume that this permutation keeps
the 0 at the last position. Since \((a_i + 1)k\) is not divisible by 11 for any \(i\),
there is a unique \(b_i \in \{0, 1, \ldots, 9\}\) such that \(b_i \equiv (a_i + 1)k - 1 \pmod{11}\).
Hence the number \(\overline{b_{2m-1}...b_1}\) belongs to \(B\).
Thus for fixed \(a_0 \in \{0, 1, 2, \ldots, 9\}\), to each \(a \in A\) such that the last
digit of \(a\) is \(a_0\) we associate a unique \(b \in B\). Conversely, having \(a_0 \in \{0, 1, 2, \ldots, 9\}\) fixed, from any number \(\overline{b_{2m-1}...b_1} \in B\) we can reconstruct
\(\overline{a_{2m-1}...a_1a_0} \in A\). Hence \(|A| = |B|\), i.e., \(f(2m) = 10f(2m - 1)\).

13. Denote by \(K\) and \(L\) the intersec-
tions of the bisectors of \(\angle ABC\) and
\(\angle ADC\) with the line \(AC\), respec-
tively. Since \(AB : BC = AK : KC\)
and \(AD : DC = AL : LC\), we have
to prove that
\[
PQ = QR \iff \frac{AB}{BC} = \frac{AD}{DC}. \tag{1}
\]
Since the quadrilaterals \(AQDR\) and
\(QPCD\) are cyclic, we see that
\[
\angle RDQ = \angle BAC \text{ and } \angle QDP = \angle ACB.
\]
By the law of sines it follows that
\[
\frac{AB}{BC} = \frac{\sin(\angle ACB)}{\sin(\angle BAC)} \text{ and that } QR = AD\sin(\angle RDQ), \quad QP = CD\sin(\angle QDP).
\]
Now we have
\[\frac{AB}{BC} = \frac{\sin(\angle ACB)}{\sin(\angle BAC)} = \frac{\sin(\angle QDP)}{\sin(\angle RDQ)} = \frac{AD \cdot QP}{QR \cdot CD}.\]

The statement (1) follows directly.

14. Denote by \( R \) the intersection point of the bisector of \( \angle AQC \) and the line \( AC \). From \( \triangle ACQ \) we get

\[\frac{AR}{RC} = \frac{AQ}{QC} = \frac{\sin \angle QCA}{\sin \angle QAC}.\]

By the sine version of Ceva’s theorem we have \( \frac{\sin \angle APB}{\sin \angle BPC} = \frac{\sin \angle QAC}{\sin \angle QCP} \), \( \frac{\sin \angle QCP}{\sin \angle QCA} = 1 \), which is equivalent to

\[\frac{\sin \angle APB}{\sin \angle BPC} = \left(\frac{\sin \angle QCA}{\sin \angle QAC}\right)^2\]

because \( \angle QCA = \angle PAQ \) and \( \angle QAC = \angle QCP \). Denote by \( S(XYZ) \) the area of a triangle \( XYZ \). Then

\[\frac{\sin \angle APB}{\sin \angle BPC} = \frac{AP \cdot BP \cdot \sin \angle APB}{BP \cdot CP \cdot \sin \angle BPC} = \frac{S(\Delta ABP)}{S(\Delta BCP)} = \frac{AB}{BC},\]

which implies that \( \left(\frac{AR}{RC}\right)^2 = \frac{AB}{BC} \). Hence \( R \) does not depend on \( \Gamma \).

15. From the given equality we see that \( 0 = (BP^2 + PE^2) - (CP^2 + PF^2) = BF^2 - CE^2 \), so \( BF = CE = x \) for some \( x \). Similarly, there are \( y \) and \( z \) such that \( CD = AF = y \) and \( BD = AE = z \). It is easy to verify that \( D, E, \) and \( F \) must lie on the segments \( BC, CA, AB \).

Denote by \( a, b, c \) the length of the segments \( BC, CA, AB \). It follows that \( a = z + y, b = z + x, c = x + y \), so \( D, E, F \) are the points where the excircles touch the sides of \( \triangle ABC \). Hence \( P, D, \) and \( I_A \) are collinear and

\[\angle PI_A C = \angle DI_A C = 90^\circ - \frac{180^\circ - \angle ACB}{2} = \frac{\angle ACB}{2}.\]

In the same way we obtain that \( \angle PI_B C = \frac{\angle ACB}{2} \) and \( PI_B = PI_A \).

Analogously, we get \( PI_C = PI_B \), which implies that \( P \) is the circumcenter of the triangle \( I_AI_BI_C \).

16. Apply an inversion with center at \( P \) and radius \( r \); let \( \hat{X} \) denote the image of \( X \). The circles \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \) are transformed into lines \( \hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_3, \hat{\Gamma}_4 \), where \( \hat{\Gamma}_1 \parallel \hat{\Gamma}_3 \) and \( \hat{\Gamma}_2 \parallel \hat{\Gamma}_4 \), and therefore \( \hat{A}\hat{B}\hat{C}\hat{D} \) is a parallelogram.

Further, we have \( AB = \frac{r^2}{P_A \cdot P_B} \hat{A}\hat{B}, BC = \frac{r^2}{P_B \cdot P_C} \hat{B}\hat{C}, CD = \frac{r^2}{P_C \cdot P_D} \hat{C}\hat{D}, DA = \frac{r^2}{P_D \cdot P_A} \hat{D}\hat{A} \) and \( PB = \frac{r^2}{P_B^2} \hat{A}\hat{B} \), \( PD = \frac{r^2}{P_D} \hat{A}\hat{D} \). The equality to be proven becomes

\[\frac{P\hat{D}^2}{P\hat{B}^2} : \frac{\hat{A}\hat{B} \cdot \hat{B}\hat{C}}{\hat{A}\hat{D} \cdot \hat{D}\hat{C}} = \frac{P\hat{D}^2}{P\hat{B}^2},\]

which holds because \( \hat{A}\hat{B} = \hat{C}\hat{D} \) and \( \hat{B}\hat{C} = \hat{D}\hat{A} \).
17. The triangles $PDE$ and $CFG$ are homothetic; hence lines $FD$, $GE$, and $CP$ intersect at one point. Let $Q$ be the intersection point of the line $CP$ and the circumcircle of $\triangle ABC$. The required statement will follow if we show that $Q$ lies on the lines $GE$ and $FD$.

Since $\angle CFG = \angle CBA = \angle CQA$, the quadrilateral $AQPF$ is cyclic. Analogously, $BQPG$ is cyclic. However, the isosceles trapezoid $BDPG$ is also cyclic; it follows that $B,Q,D,P,G$ lie on a circle. Therefore we get

$$\angle PQF = \angle PAC, \; \angle PQD = \angle PBA. \quad (1)$$

Since $I$ is the incenter of $\triangle ABC$, we have $\angle CAI = \frac{1}{2} \angle CAB = \frac{1}{2} \angle CBA = \angle IBA$; hence $CA$ is the tangent at $A$ to the circumcircle of $\triangle ABI$. This implies that $\angle PAC = \angle PBA$, and it follows from (1) that $\angle PQF = \angle PQD$, i.e., that $F,D,Q$ are also collinear. Similarly, $G,E,Q$ are collinear and the claim is thus proved.

18. Let $ABCDEF$ be the given hexagon. We shall use the following lemma.

**Lemma.** If $\angle XYZ \geq 60^\circ$ and if $M$ is the midpoint of $XY$, then $MZ \leq \frac{\sqrt{3}}{2} XY$, with equality if and only if $\triangle XYZ$ is equilateral.

**Proof.** Let $Z'$ be the point such that $\triangle XYZ'$ is equilateral. Then $Z$ is inside the circle circumscribed about $\triangle XYZ'$. Consequently $MZ \leq MZ' = \frac{\sqrt{3}}{2} XY$, with equality if and only if $Z = Z'$.

Set $AD \cap BE = P$, $BE \cap CF = Q$, and $CF \cap AD = R$. Suppose $\angle APB = \angle DPE > 60^\circ$, and let $K,L$ be the midpoints of the segments $AB$ and $DE$ respectively. Then by the lemma,

$$\frac{\sqrt{3}}{2} (AB + DE) = KL \leq PK + PL < \frac{\sqrt{3}}{2} (AB + DE),$$

which is impossible. Therefore $\angle APB \leq 60^\circ$ and similarly $\angle BQC \leq 60^\circ$, $\angle CRD \leq 60^\circ$. But the sum of the angles $\angle APB, \angle BQC, \angle CRD$ is $180^\circ$, from which we conclude that these angles are all equal to $60^\circ$, and moreover that the triangles $\triangle APB, \triangle BQC, \triangle CRD$ are equilateral. Thus $\angle ABC = \angle ABP + \angle QBC = 120^\circ$, and in the same way all angles of the hexagon are equal to $120^\circ$.

19. Let $D, E, F$ be the midpoints of $BC, CA, AB$, respectively. We construct smaller semicircles $\Gamma_d, \Gamma_e, \Gamma_f$ inside $\triangle ABC$ with centers $D, E, F$ and radii $d = \frac{s-a}{2}, \; e = \frac{s-b}{2}, \; f = \frac{s-c}{2}$ respectively. Since $DE = d + e, \; DF = d + f$, and $EF = e + f$, we deduce that $\Gamma_d, \Gamma_e, \Gamma_f$ touch each other at the points $D_1, E_1, F_1$ of tangency of the incircle $\gamma$ of $\triangle DEF$ with its sides ($D_1 \in EF$, etc.). Consider the circle $\Gamma_g$ with center $O$ and radius $g$ that lies inside $\triangle DEF$ and tangents $\Gamma_d, \Gamma_e, \Gamma_f$. 
Now let $OD, OE, OF$ meet the semicircles $\Gamma_d, \Gamma_e, \Gamma_f$ at $D', E', F'$ respectively. We have $OD' = OD + DD' = g + d + \frac{s}{2} = g + \frac{t}{2}$ and similarly $OE' = OF' = g + \frac{t}{2}$. It follows that the circle with center $O$ and radius $g + \frac{t}{2}$ touches all three semicircles, and consequently $t = g + \frac{t}{2} > \frac{t}{2}$. Now set the coordinate system such that we have the points $D_1(0, 0), E(-e, 0), F(f, 0)$ and such that the $y$ coordinate of $D$ is positive.

Apply the inversion with center $D_1$ and unit radius. This inversion maps the circles $\Gamma_e$ and $\Gamma_f$ to the lines $\widehat{\Gamma}_e \left[ x = -\frac{1}{2e} \right]$ and $\widehat{\Gamma}_e \left[ x = \frac{1}{2f} \right]$ respectively, and the circle $\gamma$ goes to the line $\widehat{\gamma} \left[ y = \frac{1}{3} \right]$. The images $\widehat{\Gamma}_d$ and $\widehat{\Gamma}_g$ of $\Gamma_d, \Gamma_g$ are the circles that touch the lines $\widehat{\Gamma}_e$ and $\widehat{\Gamma}_f$. Since $\widehat{\Gamma}_d, \widehat{\Gamma}_g$ are perpendicular to $\gamma$, they have radii equal to $R = \frac{1}{4e} + \frac{1}{4f}$ and centers at $\left( -\frac{1}{4e} + \frac{1}{4f}, \frac{1}{3} \right)$ and $\left( -\frac{1}{4e} + \frac{1}{4f}, \frac{1}{3} + 2R \right)$ respectively. Let $p$ and $P$ be the distances from $D_1(0, 0)$ to the centers of $\Gamma_g$ and $\widehat{\Gamma}_g$ respectively. We have that $P^2 = \left( \frac{1}{4e} - \frac{1}{4f} \right)^2 + \left( \frac{1}{3} + 2R \right)^2$, and that the circles $\Gamma_g$ and $\widehat{\Gamma}_g$ are homothetic with center of homothety $D_1$; hence $p/P = g/R$. On the other hand, $\widehat{\Gamma}_g$ is the image of $\Gamma_g$ under inversion; hence the product of the tangents from $D_1$ to these two circles is equal to 1. In other words, we obtain \( \sqrt{p^2 - g^2} \cdot \sqrt{P^2 - R^2} = 1 \). Using the relation $p/P = g/R$ we get $g = \frac{r}{P^2 - R^2}$.

The inequality we have to prove is equivalent to $(4 + 2\sqrt{3})g \leq r$. This can be proved as follows:

\[
 r - (4 + 2\sqrt{3})g = \frac{r(P^2 - R^2 - (4 + 2\sqrt{3})R/r)}{P^2 - R^2} = \frac{r}{{P^2 - R^2}} \left( \left( \frac{1}{r} + 2R \right)^2 + \left( \frac{1}{4e} - \frac{1}{4f} \right)^2 - R^2 - (4 + 2\sqrt{3}) \frac{R}{r} \right) \geq 0.
\]

Remark. One can obtain a symmetric formula for $g$:

\[
\frac{1}{2g} = \frac{1}{s - a} + \frac{1}{s - b} + \frac{1}{s - c} + \frac{2}{r}.
\]

20. Let $r_i$ be the remainder when $x_i$ is divided by $m$. Since there are at most $m^m$ types of $m$-consecutive blocks in the sequence $(r_i)$, some type will
repeat at least twice. Then since the entire sequence is determined by one
$m$-consecutive block, the entire sequence will be periodic.

The formula works both forward and backward; hence using the rule $x_i =
\sum_{j=1}^{m-1} x_{i+j}$ we can define $x_{-1}, x_{-2}, \ldots$. Thus we obtain that

$$(r_{-m}, \ldots, r_{-1}) = (0, 0, \ldots, 0, 1).$$

Hence there are $m - 1$ consecutive terms in the sequence $(x_i)$ that are
divisible by $m$.

If there were $m$ consecutive terms in the sequence $(x_i)$ divisible by $m$,
then by the recurrence relation all the terms of $(x_i)$ would be divisible by
$m$, which is impossible.

21. Let $a$ be a positive integer for which $d(a) = a^2$. Suppose that $a$ has $n + 1$
digits, $n \geq 0$. Denote by $s$ the last digit of $a$ and by $f$ the first digit of $c$.
Then $a = \ldots s$, where $\ast$ stands for a digit that is not important to us at
the moment. We have \(\ast \ldots \ast s^2 = a^2 = d = \ast \ldots \ast f \) and $b^2 = s \ast \ldots \ast =
c = f \ast \ldots \ast$.

We cannot have $s = 0$, since otherwise $c$ would have at most $2n$ digits,
while $a^2$ has either $2n + 1$ or $2n + 2$ digits. The following table gives all
possibilities for $s$ and $f$:

<table>
<thead>
<tr>
<th>$s$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>last digit of $\ast \ldots \ast s^2$</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>first digit of $s \ast \ldots \ast \ast$</td>
<td>1, 2, 3, 4, 8, 9, 1, 2, 2, 3, 4, 5, 6, 6, 7, 8, 9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We obtain from the table that $s \in \{1, 2, 3\}$ and $f = s^2$, and consequently
$c = b^2$ and $d$ have exactly $2n + 1$ digits each. Put $a = 10x + s$, where
$x < 10^n$. Then $b = 10^n s + x$, $c = 10^{2n} s^2 + 2 \cdot 10^n s x + x^2$, and $d =
2 \cdot 10^{n+1} s x + 10^2 x^2 + s^2$, so from $d = a^2$ it follows that $x = 2s \cdot \frac{10^n - b}{y}$. Thus
$a = 6 \ldots 63$, $a = 4 \ldots 42$ or $a = 2 \ldots 21$. For $n \geq 1$ we see that $a$ cannot
be $a = 6 \ldots 63$ or $a = 4 \ldots 42$ (otherwise $a^2$ would have $2n + 2$ digits).
Therefore $a$ equals $1, 2, 3$ or $2 \ldots 21$ for $n \geq 0$. It is easy to verify that
these numbers have the required property.

22. Let $a$ and $b$ be positive integers for which $\frac{a^2}{2ab^2 - b^3 + 1} = k$ is a positive
integer. Since $k > 0$, it follows that $2ab^2 > b^3$, so $2a \geq b$. If $2a > b$, then
from $2ab^2 - b^3 + 1 > 0$ we see that $a^2 > b^2(2a - b) + 1 > b^2$, i.e. $a > b$.
Therefore, if $a \leq b$, then $a = b/2$.

We can rewrite the given equation as a quadratic equation in $a$, $a^2 -
2kb^2 a + k(b^3 - 1) = 0$, which has two solutions, say $a_1$ and $a_2$, one of
which is in $\mathbb{N}_0$. From $a_1 + a_2 = 2kb^2$ and $a_1 a_2 = k(b^3 - 1)$ it follows
that the other solution is also in $\mathbb{N}_0$. Suppose w.l.o.g. that $a_1 \geq a_2$. Then
$a_1 \geq kb^2$ and

$$0 \leq a_2 = \frac{k(b^3 - 1)}{a_1} \leq \frac{k(b^3 - 1)}{kb^2} < b.$$
By the above considerations we have either $a_2 = 0$ or $a_2 = b/2$. If $a_2 = 0$, then $b^3 - 1 = 0$ and hence $a_1 = 2k$, $b = 1$. If $a_2 = b/2$, then $b = 2t$ for some $t$, and $k = b^2/4$, $a_1 = b^4/2 - b/2$. Therefore the only solutions are

$$(a, b) \in \{(2t, 1), (t, 2t), (8t^4 - t, 2t) \mid t \in \mathbb{N}\}.$$

It is easy to show that all of these pairs satisfy the given condition.

23. Assume that $b \geq 6$ has the required property. Consider the sequence $y_n = (b - 1)x_n$. From the definition of $x_n$ we easily find that $y_n = b^{2n} + b^{n+1} + 3b - 5$. Then $y_ny_{n+1} = (b - 1)^2x_nx_{n+1}$ is a perfect square for all $n > M$. Also, straightforward calculation implies

$$\left(\frac{b^{2n+1} + b^{n+2} + b^{n+1}}{2} - b^3\right)^2 < y_ny_{n+1} < \left(\frac{b^{2n+1} + b^{n+2} + b^{n+1}}{2} + b^3\right)^2.$$ 

Hence for every $n > M$ there is an integer $a_n$ such that $|a_n| < b^3$ and

$$y_ny_{n+1} = \left(\frac{b^{2n} + b^{n+1} + 3b - 5}{b^{2n+2} + b^{n+2} + 3b - 5}\right) = \left(\frac{b^{2n+1} + b^{n+1} + b + 1}{2} + a_n\right)^2.$$  

Now considering this equation modulo $b^n$ we obtain $(3b - 5)^2 \equiv a_n^2$, so that assuming that $n > 3$ we get $a_n = \pm(3b - 5)$.

If $a_n = 3b - 5$, then substituting in (1) yields $\frac{1}{4}b^{2n}(b^4 - 14b^3 + 45b^2 - 52b + 20) = 0$, with the unique positive integer solution $b = 10$. Also, if $a_n = -3b + 5$, we similarly obtain $\frac{1}{4}b^{2n}(b^4 - 14b^3 - 3b^2 + 28b + 20) - 2b^{n+1}(3b^2 - 2b - 5) = 0$ for each $n$, which is impossible.

For $b = 10$ it is easy to show that $x_n = \left(\frac{10n + 5}{3}\right)^2$ for all $n$. This proves the statement.

**Second solution.** In problems of this type, computing $z_n = \sqrt{x_n}$ asymptotically usually works.

From $\lim_{n \to \infty} \frac{b^{2n}}{(b - 1)x_n} = 1$ we infer that $\lim_{n \to \infty} \frac{b^n}{z_n} = \sqrt{b - 1}$. Furthermore, from $(bz_n + z_{n+1})(bz_n - z_{n+1}) = b^2x_n - x_{n+1} = b^{n+2} + 3b^2 - 2b - 5$ we obtain

$$\lim_{n \to \infty} (bz_n - z_{n+1}) = \frac{b\sqrt{b - 1}}{2}.$$ 

Since the $z_n$’s are integers for all $n \geq M$, we conclude that $bz_n - z_{n+1} = \frac{b\sqrt{b - 1}}{2}$ for all $n$ sufficiently large. Hence $b - 1$ is a perfect square, and moreover $b$ divides $2z_{n+1}$ for all large $n$. It follows that $b \mid 10$; hence the only possibility is $b = 10$.

24. Suppose that $m = u + v + w$ where $u, v, w$ are good integers whose product is a perfect square of an odd integer. Since $uvw$ is an odd perfect square, we have that $uvw \equiv 1 \pmod{4}$. Thus either two or none of the numbers
are congruent to 3 modulo 4. In both cases \( u + v + w \equiv 3 \) (mod 4).

Hence \( m \equiv 3 \) (mod 4).

Now we shall prove the converse: every \( m \equiv 3 \) (mod 4) has infinitely many representations of the desired type. Let \( m = 4k + 3 \). We shall represent \( m \) in the form

\[
4k + 3 = xy + yz + zx, \quad \text{for } x, y, z \text{ odd. (1)}
\]

The product of the summands is an odd square. Set \( x = 1 + 2l \) and \( y = 1 - 2l \). In order to satisfy (1), \( z \) must satisfy \( z = 2l^2 + 2k + 1 \). The summands \( xy, yz, zx \) are distinct except for finitely many \( l \), so it remains to prove that for infinitely many integers \( l \), \( |xy|, |yz|, \) and \( |zx| \) are not perfect squares. First, observe that \( |xy| = 4l^2 - 1 \) is not a perfect square for any \( l \neq 0 \).

Let \( p, q > m \) be fixed different prime numbers. The system of congruences

\[
1 + 2l \equiv p \pmod{p^2} \quad \text{and} \quad 1 - 2l \equiv q \pmod{q^2}
\]

has infinitely many solutions \( l \) by the Chinese remainder theorem. For any such \( l \), the number \( z = 2l^2 + 2k + 1 \) is divisible by neither \( p \) nor \( q \), and hence \( |xz| \) (respectively \( |yz| \)) is divisible by \( p \), but not by \( p^2 \) (respectively by \( q \), but not by \( q^2 \)). Thus \( xz \) and \( yz \) are also good numbers.

25. Suppose that for every prime \( q \), there exists an \( n \) for which \( n^p \equiv p \pmod{q} \). Assume that \( q = kp + 1 \). By Fermat’s theorem we deduce that \( p^k \equiv n^{kp} = n^{q-1} \equiv 1 \pmod{q} \), so \( q \mid p^k - 1 \).

It is known that any prime \( q \) such that \( q \mid \frac{p^{q-1}-1}{p-1} \) must satisfy \( q \equiv 1 \pmod{p} \). Indeed, from \( q \mid p^{q-1} - 1 \) it follows that \( q \mid p^{\gcd(p,q-1)} - 1 \); but \( q \nmid p - 1 \) because \( \frac{p^{q-1}-1}{p-1} \equiv 1 \pmod{p-1} \), so \( \gcd(p,q-1) \neq 1 \). Hence \( \gcd(p,q-1) = p \).

Now suppose \( q \) is any prime divisor of \( \frac{p^{q-1}-1}{p-1} \). Then \( q \mid \gcd(p^{q-1} - 1, p^p - 1) = p^{\gcd(p,q)} - 1 \), which implies that \( \gcd(p,q) > 1 \), so \( p \mid q \). Consequently \( q \equiv 1 \pmod{p^2} \). However, the number \( \frac{p^{q-1}}{p-1} = p^{p-1} + \cdots + p + 1 \) must have at least one prime divisor that is not congruent to 1 modulo \( p^2 \). Thus we arrived at a contradiction.

Remark. Taking \( q \equiv 1 \pmod{p} \) is natural, because for every other \( q \), \( n^p \) takes all possible residues modulo \( q \) (including \( p \) too). Indeed, if \( p \nmid q - 1 \), then there is an \( r \in \mathbb{N} \) satisfying \( pr \equiv 1 \pmod{q - 1} \); hence for any \( a \) the congruence \( n^p \equiv a \pmod{q} \) has the solution \( n \equiv a^r \pmod{q} \).

The statement of the problem itself is a special case of the Chebotarev’s theorem.

26. Define the sequence \( x_k \) of positive reals by \( a_k = \cosh x_k \) (cosh is the hyperbolic cosine defined by \( \cosh t = \frac{e^t + e^{-t}}{2} \)). Since \( \cosh(2x_k) = 2a_k^2 - 1 = \cosh x_{k+1} \), it follows that \( x_{k+1} = 2x_k \) and thus \( x_k = \lambda \cdot 2^k \) for some \( \lambda > 0 \). From the condition \( a_0 = 2 \) we obtain \( \lambda = \log(2 + \sqrt{3}) \). Therefore

\[
a_n = \frac{(2 + \sqrt{3})^{2^n} + (2 - \sqrt{3})^{2^n}}{2}.
\]
Let $p$ be a prime number such that $p \mid a_n$. We distinguish the following two cases:

(i) There exists an $m \in \mathbb{Z}$ such that $m^2 \equiv 3 \pmod{p}$. Then we have

$$ (2 + m)^{2^n} + (2 - m)^{2^n} \equiv 0 \pmod{p}. \quad (1) $$

Since $(2 + m)(2 - m) = 4 - m^2 \equiv 1 \pmod{p}$, multiplying both sides of $(1)$ by $(2 + m)^{2^n}$ gives $(2 + m)^{2^{n+1}} \equiv -1 \pmod{p}$. It follows that the multiplicative order of $(2 + m)$ modulo $p$ is $2^{n+2}$, or $2^{n+2} \mid p - 1$, which implies that $2^{n+3} \mid (p - 1)(p + 1) = p^2 - 1$.

(ii) $m^2 \equiv 3 \pmod{p}$ has no integer solutions. We will work in the algebraic extension $\mathbb{Z}_p(\sqrt{3})$ of the field $\mathbb{Z}_p$. In this field $\sqrt{3}$ plays the role of $m$, so as in the previous case we obtain $(2 + \sqrt{3})^{2^n+1} = -1$; i.e., the order of $2 + \sqrt{3}$ in the multiplicative group $\mathbb{Z}_p(\sqrt{3})^*$ is $2^{n+2}$. We cannot finish the proof as in the previous case: in fact, we would conclude only that $2^{n+2}$ divides the order $p^2 - 1$ of the group. However, it will be enough to find an $u \in \mathbb{Z}_p(\sqrt{3})$ such that $u^2 = 2 + \sqrt{3}$, since then the order of $u$ is equal to $2^{n+3}$.

Note that $(1 + \sqrt{3})^2 = 2(2 + \sqrt{3})$. Thus it is sufficient to prove that $\frac{1}{2}$ is a perfect square in $\mathbb{Z}_p(\sqrt{3})$. But we know that in this field $a_n = 0 = 2a_{n-1}^2 - 1$, and hence $2a_{n-1}^2 = 1$ which implies $\frac{1}{2} = a_{n-1}^2$. This completes the proof.

27. Let $p_1, p_2, \ldots, p_r$ be distinct primes, where $r = p - 1$. Consider the sets

$$ B_i = \{p_i, p_i^{p+1}, \ldots, p_i^{(r-1)p+1}\} \quad \text{and} \quad B = \bigcup_{i=1}^r B_i. $$

Then $B$ has $(p - 1)^2$ elements and satisfies (i) and (ii).

Now suppose that $|A| \geq r^2 + 1$ and that $A$ satisfies (i) and (ii), and let $\{t_1, \ldots, t_{r^2+1}\}$ be distinct elements of $A$, where $t_j = p_1^{\alpha_{j1}} \cdot p_2^{\alpha_{j2}} \cdots p_r^{\alpha_{jr}}$. We shall show that the product of some elements of $A$ is a perfect $p$th power, i.e., that there exist $\tau_j \in \{0, 1\}$ $(1 \leq j \leq r^2 + 1)$, not all equal to 0, such that $T = t_1^{\tau_1} \cdot t_2^{\tau_2} \cdots t_{r^2+1}^{\tau_{r^2+1}}$ is a $p$th power. This is equivalent to the condition that

$$ \sum_{j=1}^{r^2+1} \alpha_{ij} \tau_j \equiv 0 \pmod{p} $$

holds for all $i = 1, \ldots, r$.

By Fermat’s theorem it is sufficient to find integers $x_1, \ldots, x_{r^2+1}$, not all zero, such that the relation

$$ \sum_{j=1}^{r^2+1} \alpha_{ij} x_j^{\tau_j} \equiv 0 \pmod{p} $$

is satisfied for all $i \in \{1, \ldots, r\}$. Set $F_i = \sum_{j=1}^{r^2+1} \alpha_{ij} x_j^{\tau_j}$. We want to find $x_1, \ldots, x_r$ such that $F_1 \equiv F_2 \equiv \cdots \equiv F_r \equiv 0 \pmod{p}$, which is by Fermat’s theorem equivalent to
\[ F(x_1, \ldots, x_r) = F'_1 + F'_2 + \cdots + F'_r \equiv 0 \pmod{p}. \] (1)

Of course, one solution of (1) is \((0, \ldots, 0)\): we are not satisfied with it because it generates the empty subset of \(A\), but it tells us that (1) has at least one solution.

We shall prove that the number of solutions of (1) is divisible by \(p\), which will imply the existence of a nontrivial solution and thus complete the proof. To do this, consider the sum \( \sum F(x_1, \ldots, x_{r^2+1})^r \) taken over all vectors \((x_1, \ldots, x_{r^2+1})\) in the vector space \(\mathbb{Z}_p^{r^2+1}\). Our statement is equivalent to

\[ \sum F(x_1, \ldots, x_{r^2+1})^r \equiv 0 \pmod{p}. \] (2)

Since the degree of \(F^r\) is \(r^2\), in each monomial in \(F^r\) at least one of the variables is missing. Consider any of these monomials, say \(b x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}\). Then the sum \( \sum b x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} \), taken over the set of all vectors \((x_1, \ldots, x_{r^2+1}) \in \mathbb{Z}_p^{r^2+1}\), is equal to

\[ p^{r^2+1-u} \sum_{(x_1, \ldots, x_k) \in \mathbb{Z}_p^k} b x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}, \]

which is divisible by \(p\), so that (2) is proved. Thus the answer is \((p-1)^2\).
4.45 Solutions to the Shortlisted Problems of IMO 2004

1. By symmetry, it is enough to prove that \( t_1 + t_2 > t_3 \). We have

\[
\left( \sum_{i=1}^{n} t_i \right) \left( \sum_{i=1}^{n} \frac{1}{t_i} \right) = n^2 + \sum_{i<j} \left( \frac{t_i}{t_j} + \frac{t_j}{t_i} - 2 \right) .
\] (1)

All the summands on the RHS are positive, and therefore the RHS is not smaller than \( n^2 + T \), where \( T = (t_1/t_3 + t_3/t_1 - 2) + (t_2/t_3 + t_3/t_2 - 2) \).

We note that \( T \) is increasing as a function in \( t_3 \) for \( t_3 \geq \max\{t_1, t_2\} \).

If \( t_1 + t_2 = t_3 \), then \( T = (t_1 + t_2)(1/t_1 + 1/t_2) - 1 \geq 3 \) by the Cauchy–Schwarz inequality. Hence, if \( t_1 + t_2 < t_3 \), we have \( T \geq 1 \), and consequently the RHS in (1) is greater than or equal to \( n^2 + 1 \), a contradiction.

Remark. In can be proved, for example using Lagrange multipliers, that if \( n^2 + 1 \) in the problem is replaced by \((n + \sqrt{10} - 3)^2\), then the statement remains true. This estimate is the best possible.

2. We claim that the sequence \( \{a_n\} \) must be unbounded.

The condition of the sequence is equivalent to \( a_n > 0 \) and \( a_{n+1} = a_n + a_{n-1} \) or \( a_n - a_{n-1} \). In particular, if \( a_n < a_{n-1} \), then \( a_{n+1} > \max\{a_n, a_{n-1}\} \).

Let us remove all \( a_n \) such that \( a_n < a_{n-1} \). The obtained sequence \( (b_m)_{m \in \mathbb{N}} \) is strictly increasing. Thus the statement of the problem will follow if we prove that \( b_{m+1} - b_m \geq b_m - b_{m-1} \) for all \( m \geq 2 \).

Let \( b_{m+1} = a_{n+2} \) for some \( n \). Then \( a_{n+2} > a_{n+1} \). We distinguish two cases:

(i) If \( a_{n+1} > a_n \), we have \( b_m = a_{n+1} \) and \( b_{m+1} - b_m \geq a_{n-1} \) (since \( b_{m-1} \) is either \( a_{n-1} \) or \( a_n \)). Then \( b_{m+1} - b_m = a_{n+2} - a_{n+1} = a_n = a_{n+1} - a_{n-1} = b_m - a_{n-1} \geq b_m - b_{m-1} \).

(ii) If \( a_{n+1} < a_n \), we have \( b_m = a_n \) and \( b_{m+1} \geq a_{n} \). Consequently, \( b_{m+1} - b_m = a_{n+2} - a_n = a_{n+1} - a_{n-1} = b_m - a_{n-1} \geq b_m - b_{m-1} \).

3. The answer is yes. Every rational number \( x > 0 \) can be uniquely expressed as a continued fraction of the form \( a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} \) (where \( a_0 \in \mathbb{N} \), \( a_1, \ldots, a_n \in \mathbb{N} \)). Then we write \( x = [a_0; a_1, a_2, \ldots, a_n] \).

Since \( n \) depends only on \( x \), the function \( s(x) = (-1)^n \) is well-defined. For \( x < 0 \) we define \( s(x) = -s(-x) \), and set \( s(0) = 1 \). We claim that this \( s(x) \) satisfies the requirements of the problem.

The equality \( s(x)s(y) = -1 \) trivially holds if \( x + y = 0 \).

Suppose that \( xy = 1 \). We may assume w.l.o.g. that \( x > y > 0 \). Then \( x > 1 \), so if \( x = [a_0; a_1, a_2, \ldots, a_n] \), then \( a_0 \geq 1 \) and \( y = 0 + 1/x = [0; a_0, a_1, a_2, \ldots, a_n] \). It follows that \( s(x) = (-1)^n \), \( s(y) = (-1)^{n+1} \), and hence \( s(x)s(y) = -1 \).

Finally, suppose that \( x + y = 1 \). We consider two cases:

(i) Let \( x, y > 0 \) We may assume w.l.o.g. that \( x > 1/2 \). Then there exist natural numbers \( a_2, \ldots, a_n \) such that \( x = [0; a_2, \ldots, a_n] = 1/(1 + 1/t) \), where \( t = [a_2, \ldots, a_n] \). Since \( y = 1 - x = 1/(1 + t) = \)
\[0; 1 + a_2, a_3, \ldots, a_n], \text{ we have } s(x) = (-1)^n \text{ and } s(y) = (-1)^{n-1}, \]
giving us \(s(x)s(y) = -1\).

(ii) Let \(x > 0 > y\). If \(a_0, \ldots, a_n \in \mathbb{N}\) are such that \(-y = [a_0; a_1, \ldots, a_n]\),
then \(x = [1 + a_0; a_1, \ldots, a_n]\). Thus \(s(y) = -s(-y) = -(-1)^n\) and
\(s(x) = (-1)^n\), so again \(s(x)s(y) = -1\).

4. Let \(P(x) = a_0 + a_1 x + \cdots + a_n x^n\). For every \(x \in \mathbb{R}\) the triple \((a, b, c) =
(6x, 3x, -2x)\) satisfies the condition \(ab + bc + ca = 0\). Then the condition
on \(P\) gives us \(P(3x) + P(5x) + P(-8x) = 2P(7x)\) for all \(x\), implying that
for all \(i = 0, 1, 2, \ldots, n\) the following equality holds:
\[
(3^i + 5^i + (-8)^i - 2 \cdot 7^i) a_i = 0.
\]
Suppose that \(a_i \neq 0\). Then \(K(i) = 3^i + 5^i + (-8)^i - 2 \cdot 7^i = 0\). But \(K(i)\) is
negative for \(i\) odd and positive for \(i = 0\) or \(i \geq 6\) even. Only for \(i = 2\) and
\(i = 4\) do we have \(K(i) = 0\). It follows that \(P(x) = a_2 x^2 + a_4 x^4\) for some
real numbers \(a_2, a_4\).
It is easily verified that all such \(P(x)\) satisfy the required condition.

5. By the general mean inequality \((M_1 \leq M_3)\), the LHS of the inequality to
be proved does not exceed
\[
E = \frac{3}{\sqrt[3]{3}} \sqrt[3]{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 6(a + b + c)}.
\]
From \(ab + bc + ca = 1\) we obtain that \(3abc(a + b + c) = 3(ab \cdot ac +
ab \cdot bc + ac \cdot bc) \leq (ab + ac + bc)^2 = 1\); hence \(6(a + b + c) \leq \frac{2}{abc}\). Since
\[
\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{abc + bc + ca}{abc} = \frac{1}{abc},
\]
it follows that
\[
E \leq \frac{3}{\sqrt[3]{3}} \sqrt[3]{\frac{3}{abc}} \leq \frac{1}{abc},
\]
where the last inequality follows from the AM–GM inequality \(1 = ab + bc +
ca \geq 3\sqrt[3]{(abc)^2}\), i.e., \(abc \leq 1/(3\sqrt{3})\). The desired inequality now follows.
Equality holds if and only if \(a = b = c = 1/\sqrt{3}\).

6. Let us make the substitution \(z = x + y, \ t = xy\). Given \(z, t \in \mathbb{R}, \ x, y\) are
real if and only if \(4t \leq z^2\). Define \(g(x) = 2(f(x) - x)\). Now the given
functional equation transforms into
\[
f(z^2 + g(t)) = (f(z))^2 \quad \text{for all } t, z \in \mathbb{R} \text{ with } z^2 \geq 4t. \tag{1}
\]
Let us set \(c = g(0) = 2f(0)\). Substituting \(t = 0\) into (1) gives us
\[
f(z^2 + c) = (f(z))^2 \quad \text{for all } z \in \mathbb{R}. \tag{2}
\]
If \(c < 0\), then taking \(z\) such that \(z^2 + c = 0\), we obtain from (2) that
\(f(z)^2 = c/2\), which is impossible; hence \(c \geq 0\). We also observe that
\[ x > c \quad \text{implies} \quad f(x) \geq 0. \quad (3) \]

If \( g \) is a constant function, we easily find that \( c = 0 \) and therefore \( f(x) = x \), which is indeed a solution.

Suppose \( g \) is nonconstant, and let \( a, b \in \mathbb{R} \) be such that \( g(a) - g(b) = d > 0 \).

For some sufficiently large \( K \) and each \( u, v \geq K \) with \( v^2 - u^2 = d \) the equality \( u^2 + g(a) = v^2 + g(b) \) by (1) and (3) implies \( f(u) = f(v) \). This further leads to \( g(u) - g(v) = 2(v - u) = \frac{d}{u + \sqrt{u^2 + d}} \). Therefore every value from some suitably chosen segment \( [\delta, 2\delta] \) can be expressed as \( g(u) - g(v) \), with \( u \) and \( v \) bounded from above by some \( M \).

Consider any \( x, y \) with \( y > x \geq 2\sqrt{M} \) and \( \delta < y^2 - x^2 < 2\delta \). By the above considerations, there exist \( u, v \leq M \) such that \( g(u) - g(v) = y^2 - x^2 \), i.e., \( x^2 + g(u) = y^2 + g(v) \). Since \( x^2 \geq 4u \) and \( y^2 \geq 4v \), (1) leads to \( f(x)^2 = f(y)^2 \). Moreover, if we assume w.l.o.g. that \( 4M \geq c^2 \), we conclude from (3) that \( f(x) = f(y) \). Since this holds for any \( x, y \geq 2\sqrt{M} \) with \( y^2 - x^2 \in [\delta, 2\delta] \), it follows that \( f(x) \) is eventually constant, say \( f(x) = k \) for \( x \geq N = 2\sqrt{M} \). Setting \( x > N \) in (2) we obtain \( k^2 = k \), so \( k = 0 \) or \( k = 1 \).

By (2) we have \( f(-z) = \pm f(z) \), and thus \( |f(z)| \leq 1 \) for all \( z \leq -N \). Hence \( g(u) = 2f(u) - 2u \geq -2 - 2u \) for \( u \leq -N \), which implies that \( g \) is unbounded. Hence for each \( z \) there exists \( t \) such that \( z^2 + g(t) > N \), and consequently \( f(z)^2 = f(z^2 + g(t)) = k^2 \). Therefore \( f(z) = \pm k \) for each \( z \).

If \( k = 0 \), then \( f(x) \equiv 0 \), which is clearly a solution. Assume \( k = 1 \). Then \( c = 2f(0) = 2 \) (because \( c \geq 0 \)), which together with (3) implies \( f(x) = 1 \) for all \( x \geq 2 \). Suppose that \( f(t) = -1 \) for some \( t < 2 \). Then \( t - g(t) = 3t + 2 > 4t \). If also \( t - g(t) \geq 0 \), then for some \( z \in \mathbb{R} \) we have \( z^2 = t - g(t) > 4t \), which by (1) leads to \( f(z)^2 = f(z^2 + g(t)) = f(t) = -1 \), which is impossible. Hence \( t - g(t) < 0 \), giving us \( t < -2/3 \). On the other hand, if \( X \) is any subset of \((-\infty, -2/3)\), the function \( f \) defined by \( f(x) = -1 \) for \( x \in X \) and \( f(x) = 1 \) satisfies the requirements of the problem.

To sum up, the solutions are \( f(x) = x \), \( f(x) = 0 \) and all functions of the form

\[ f(x) = \begin{cases} 1, & x \notin X, \\ -1, & x \in X, \end{cases} \]

where \( X \subset (-\infty, -2/3) \).

7. Let us set \( c_k = A_{k-1}/A_k \) for \( k = 1, 2, \ldots, n \), where we define \( A_0 = 0 \). We observe that \( a_k/A_k = (kA_k - (k-1)A_{k-1})/A_k = k - (k-1)c_k \). Now we can write the LHS of the inequality to be proved in terms of \( c_k \), as follows:

\[ \sqrt[n]{\frac{G_n}{A_n}} = n^{2/3}c_2^2 \cdots c_{n-1} \quad \text{and} \quad \frac{g_n}{G_n} = \sqrt[n]{\prod_{k=1}^n (k - (k-1)c_k)}. \]
By the AM–GM inequality we have
\[
\frac{n^2}{1^n(n+1)/2c_2c_3 \ldots c_{n-1}} \leq \frac{1}{n} \left( \frac{n(n+1)}{2} + \sum_{k=2}^{n} (k-1)c_k \right)
\]
\[
= \frac{n+1}{2} + \frac{1}{n} \sum_{k=1}^{n} (k-1)c_k.
\] (1)

Also by the AM–GM inequality, we have
\[
\sqrt[n]{\prod_{k=1}^{n} (k - (k-1)c_k)} \leq \frac{n+1}{2} - \frac{1}{n} \sum_{k=1}^{n} (k-1)c_k.
\] (2)

Adding (1) and (2), we obtain the desired inequality. Equality holds if and only if \(a_1 = a_2 = \cdots = a_n\).

8. Let us write \(n = 10001\). Denote by \(T\) the set of ordered triples \((a, C, S)\), where \(a\) is a student, \(C\) a club, and \(S\) a society such that \(a \in C\) and \(C \in S\). We shall count \(|T|\) in two different ways.

Fix a student \(a\) and a society \(S\). By (ii), there is a unique club \(C\) such that \((a, C, S) \in T\). Since the ordered pair \((a, S)\) can be chosen in \(nk\) ways, we have that \(|T| = nk\).

Now fix a club \(C\). By (iii), \(C\) is in exactly \(|C|(|C| - 1)/2\) societies, so there are \(|C|(|C| - 1)/2\) triples from \(T\) with second coordinate \(C\). If \(C\) is the set of all clubs, we obtain \(|T| = \sum_{C \in \mathcal{C}} \frac{|C|(|C| - 1)}{2}\). But we also conclude from (i) that
\[
\sum_{C \in \mathcal{C}} \frac{|C|(|C| - 1)}{2} = \frac{n(n-1)}{2}.
\]

Therefore \(n(n-1)/2 = nk\), i.e., \(k = (n-1)/2 = 5000\).

On the other hand, for \(k = (n-1)/2\) there is a desired configuration with only one club \(C\) that contains all students and \(k\) identical societies with only one element (the club \(C\)). It is easy to verify that (i)–(iii) hold.

9. Obviously we must have \(2 \leq k \leq n\). We shall prove that the possible values for \(k\) and \(n\) are \(2 \leq k \leq n \leq 3\) and \(3 \leq k \leq n\). Denote all colors and circles by \(1, \ldots, n\). Let \(F(i, j)\) be the set of colors of the common points of circles \(i\) and \(j\).

Suppose that \(k = 2 < n\). Consider the ordered pairs \((i, j)\) such that color \(j\) appears on the circle \(i\). Since \(k = 2\), clearly there are exactly \(2n\) such pairs. On the other hand, each of the \(n\) colors appears on at least two circles, so there are at least \(2n\) pairs \((i, j)\), and equality holds only if each color appears on exactly 2 circles. But then at most two points receive each of the \(n\) colors and there are \(n(n-1)\) points, implying that \(n(n-1) = 2n\), i.e., \(n = 3\). It is easy to find examples for \(k = 2\) and \(n = 2\) or \(3\).

Next, let \(k = 3\). An example for \(n = 3\) is given by \(F(i, j) = \{i, j\}\) for each \(1 \leq i < j \leq 3\). Assume \(n \geq 4\). Then an example is given by \(F(1, 2) = \{1, 2\}\).
\{1, 2\}, F(i, i + 1) = \{i\} for \(i = 2, \ldots, n - 2\), \(F(n - 1, n) = \{n - 2, n - 1\}\) and \(F(i, j) = n\) for all other \(i, j > i\).

We now prove by induction on \(k\) that a desired coloring exists for each \(n \geq k \geq 3\). Let there be given \(n\) circles. By the inductive hypothesis, circles 1, 2, \ldots, \(n - 1\) can be colored in \(n - 1\) colors, \(k\) of which appear on each circle, such that color \(i\) appears on circle \(i\). Then we set \(F(i, n) = \{i, n\}\) for \(i = 1, \ldots, k\) and \(F(i, n) = \{n\}\) for \(i > n\). We thus obtain a coloring of the \(n\) circles in \(n\) colors, such that \(k + 1\) colors (including color \(i\)) appear on each circle \(i\).

10. The least number of edges of such a graph is \(n\).

We note that deleting edge \(AB\) of a 4-cycle \(ABCD\) from a connected and nonbipartite graph \(G\) yields a connected and nonbipartite graph, say \(H\). Indeed, the connectedness is obvious; also, if \(H\) were bipartite with partition of the set of vertices into \(P_1\) and \(P_2\), then w.l.o.g. \(A, C \in P_1\) and \(B, D \in P_2\), so \(G = H \cup \{AB\}\) would also be bipartite with the same partition, a contradiction.

Any graph that can be obtained from the complete \(n\)-graph in the described way is connected and has at least one cycle (otherwise it would be bipartite); hence it must have at least \(n\) edges.

Now consider a complete graph with vertices \(V_1, V_2, \ldots, V_n\). Let us remove every edge \(V_iV_j\) with \(3 \leq i < j < n\) from the cycle \(V_2V_iV_jV_n\). Then for \(i = 3, \ldots, n - 1\) we remove edges \(V_2V_i\) and \(V_iV_n\) from the cycles \(V_1V_iV_2V_n\) and \(V_1V_iV_nV_2\) respectively, thus obtaining a graph with exactly \(n\) edges:

\(V_1V_i\) \((i = 2, \ldots, n)\) and \(V_2V_n\).

11. Consider the matrix \(A = (a_{ij})_{i,j=1}^n\) such that \(a_{ij}\) is equal to 1 if \(i, j \leq n/2\), 
\(-1\) if \(i, j > n/2\), and 0 otherwise. This matrix satisfies the conditions from the problem and all row sums and column sums are equal to \(\pm n/2\). Hence \(C \geq n/2\).

Let us show that \(C = n/2\). Assume to the contrary that there is a matrix \(B = (b_{ij})_{i,j=1}^n\) all of whose row sums and column sums are either greater than \(n/2\) or smaller than \(-n/2\). We may assume w.l.o.g. that at least \(n/2\) row sums are positive and, permuting rows if necessary, that the first \(n/2\) rows have positive sums. The sum of entries in the \(n/2 \times n\) submatrix \(B'\) consisting of first \(n/2\) rows is greater than \(n^2/4\), and since each column of \(B'\) has sum at most \(n/2\), it follows that more than \(n/2\) column sums of \(B'\), and therefore also of \(B\), are positive. Again, suppose w.l.o.g. that the first \(n/2\) column sums are positive. Thus the sums \(R^+\) and \(C^+\) of entries in the first \(n/2\) rows and in the first \(n/2\) columns respectively are greater than \(n^2/4\). Now the sum of all entries of \(B\) can be written as

\[
\sum_{i>j \leq n/2} a_{ij} = R^+ + C^+ + \sum_{i \leq n/2} a_{ij} - \sum_{i> j \leq n/2} a_{ij} > \frac{n^2}{2} - \frac{n^2}{4} - \frac{n^2}{4} = 0,
\]
a contradiction. Hence \(C = n/2\), as claimed.
12. We say that a number \( n \in \{1, 2, \ldots, N\} \) is \textit{winning} if the player who is on turn has a winning strategy, and \textit{losing} otherwise. The game is of type \( A \) if and only if 1 is a losing number. Let us define \( n_0 = N, n_{i+1} = \lfloor n_i / 2 \rfloor \) for \( i = 0, 1, \ldots \) and let \( k \) be such that \( n_k = 1 \). Consider the sets \( A_i = \{n_i + 1, \ldots, n_i\} \). We call a set \( A_i \) \textit{all-winning} if all numbers from \( A_i \) are winning, \textit{even-winning} if even numbers are winning and odd are losing, and \textit{odd-winning} if odd numbers are winning and even are losing.

(i) Suppose \( A_i \) is even-winning and consider \( A_{i+1} \). Multiplying any number from \( A_{i+1} \) by 2 yields an even number from \( A_i \), which is a losing number. Thus \( x \in A_{i+1} \) is winning if and only if \( x + 1 \) is losing, i.e., if and only if it is even. Hence \( A_{i+1} \) is also even-winning.

(ii) Suppose \( A_i \) is odd-winning. Then each \( k \in A_{i+1} \) is winning, since \( 2k \) is losing. Hence \( A_{i+1} \) is all-winning.

(iii) Suppose \( A_i \) is all-winning. Multiplying \( x \in A_{i+1} \) by two is then a losing move, so \( x \) is winning if and only if \( x + 1 \) is losing. Since \( n_{i+1} \) is losing, \( A_{i+1} \) is odd-winning if \( n_{i+1} \) is even and even-winning otherwise.

We observe that \( A_0 \) is even-winning if \( N \) is odd and odd-winning otherwise. Also, if some \( A_i \) is even-winning, then all \( A_{i+1}, A_{i+2}, \ldots \) are even-winning and thus 1 is losing; i.e., the game is of type \( A \). The game is of type \( B \) if and only if the sets \( A_0, A_1, \ldots, A_{n-1} \) are alternately odd-winning and all-winning with \( A_0 \) odd-winning, which is equivalent to \( N = n_0, n_2, n_4, \ldots \) all being even. Thus \( N \) is of type \( B \) if and only if all digits at the odd positions in the binary representation of \( N \) are zeros.

Since 2004 = 11111010100 in the binary system, 2004 is of type \( A \). The least \( N > 2004 \) that is of type \( B \) is 100000000000 = 2^{11} = 2048. Thus the answer to part (b) is 2048.

13. Since \( X_i, Y_i, i = 1, \ldots, 2004 \), are 4008 distinct subsets of the set \( S_n = \{1, 2, \ldots, n\} \), it follows that \( 2^n \geq 4008 \), i.e. \( n \geq 12 \).

Suppose \( n = 12 \). Let \( \mathcal{X} = \{X_1, \ldots, X_{2004}\} \), \( \mathcal{Y} = \{Y_1, \ldots, Y_{2004}\} \), \( \mathcal{A} = \mathcal{X} \cup \mathcal{Y} \). Exactly \( 2^{12} - 4008 = 88 \) subsets of \( S_n \) do not occur in \( \mathcal{A} \).

Since each row intersects each column, we have \( X_i \cap Y_j \neq \emptyset \) for all \( i, j \). Suppose \( |X_i|, |Y_j| \leq 3 \) for some indices \( i, j \). Since then \( |X_i \cup Y_j| \leq 5 \), any of at least \( 2^2 > 88 \) subsets of \( S_n \setminus (X_i \cap Y_j) \) can occur in neither \( \mathcal{X} \) nor \( \mathcal{Y} \), which is impossible. Hence either in \( \mathcal{X} \) or in \( \mathcal{Y} \) all subsets are of size at least 4. Suppose w.l.o.g. that \( k = |X_i| = \min_i |X_i| \geq 4 \). There are

\[
n_k = \binom{12 - k}{0} + \binom{12 - k}{1} + \cdots + \binom{12 - k}{k - 1}
\]

subsets of \( S \setminus X_I \) with fewer than \( k \) elements, and none of them can be either in \( \mathcal{X} \) (because \( |X_i| \) is minimal in \( \mathcal{X} \)) or in \( \mathcal{Y} \). Hence we must have \( n_k \leq 88 \). Since \( n_4 = 93 \) and \( n_5 = 99 \), it follows that \( k \geq 6 \). But then none of the \( \binom{12}{0} + \cdots + \binom{12}{5} = 1586 \) subsets of \( S_n \) is in \( \mathcal{X} \), hence at least \( 1586 - 88 = 1498 \) of them are in \( \mathcal{Y} \). The 1498 complements of these subsets
also do not occur in $X$, which adds to 3084 subsets of $S_n$ not occurring in $X$. This is clearly a contradiction.

Now we construct a golden matrix for $n = 13$. Let

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A_m = \begin{bmatrix} A_{m-1} & A_{m-1} \\ A_{m-1} & B_{m-1} \end{bmatrix} \quad \text{for } m = 2, 3, \ldots ,$$

where $B_{m-1}$ is the $2^{m-1} \times 2^{m-1}$ matrix with all entries equal to $m + 2$. It can be easily proved by induction that each of the matrices $A_m$ is golden. Moreover, every upper-left square submatrix of $A_m$ of size greater than $2^{m-1}$ is also golden. Since $2^{10} < 2004 < 2^{11}$, we thus obtain a golden matrix of size 2004 with entries in $S_{13}$.

14. Suppose that an $m \times n$ rectangle can be covered by “hooks”. For any hook $H$ there is a unique hook $K$ that covers its “inside” square. Then also $H$ covers the inside square of $K$, so the set of hooks can be partitioned into pairs of type $\{H, K\}$, each of which forms one of the following two figures consisting of 12 squares:

Thus the $m \times n$ rectangle is covered by these tiles. It immediately follows that $12 \mid mn$.

Suppose one of $m, n$ is divisible by 4. Let w.l.o.g. $4 \mid m$. If $3 \mid n$, one can easily cover the rectangle by $3 \times 4$ rectangles and therefore by hooks. Also, if $12 \mid m$ and $n \not\in \{1, 2, 5\}$, then there exist $k, l \in \mathbb{N}_0$ such that $n = 3k + 4l$, and thus the rectangle $m \times n$ can be partitioned into $3 \times 12$ and $4 \times 12$ rectangles all of which can be covered by hooks. If $12 \mid m$ and $n = 1, 2, 5$, then it is easy to see that covering by hooks is not possible.

Now suppose that $4 \mid m$ and $4 \nmid n$. Then $m, n$ are even and the number of tiles is odd. Assume that the total number of tiles of types $A_1$ and $B_1$ is odd (otherwise the total number of tiles of types $A_2$ and $B_2$ is odd, which is analogous). If we color in black all columns whose indices are divisible by 4, we see that each tile of type $A_1$ or $B_1$ covers three black squares, which yields an odd number in total. Hence the total number of black squares covered by the tiles of types $A_2$ and $B_2$ must be odd. This is impossible, since each such tile covers two or four black squares.

15. Denote by $V_1, \ldots, V_n$ the vertices of a graph $G$ and by $E$ the set of its edges. For each $i = 1, \ldots, n$, let $A_i$ be the set of vertices connected to $V_i$ by an edge, $G_i$ the subgraph of $G$ whose set of vertices is $A_i$, and $E_i$ the set of edges of $G_i$. Also, let $v_i, e_i$, and $t_i = f(G_i)$ be the numbers of vertices, edges, and triangles in $G_i$, respectively.
The numbers of tetrahedra and triangles one of whose vertices is $V_i$ are respectively equal to $t_i$ and $e_i$. Hence

$$
\sum_{i=1}^{n} v_i = 2|E|, \quad \sum_{i=1}^{n} e_i = 3f(G) \quad \text{and} \quad \sum_{i=1}^{n} t_i = 4g(G).
$$

Since $e_i \leq v_i(v_i - 1)/2 \leq v_i^2/2$ and $e_i \leq |E|$, we obtain $e_i^2 \leq v_i^2|E|/2$, i.e., $e_i \leq v_i\sqrt{|E|}/2$. Summing over all $i$ yields $3f(G) \leq 2|E|\sqrt{|E|}/2$, or equivalently $f(G)^2 \leq 2|E|^3/9$. Since this relation holds for each graph $G_i$, it follows that

$$
t_i = f(G_i) = f(G_i)^{1/3} f(G_i)^{2/3} \leq \left(\frac{2}{9}\right)^{1/3} f(G)^{1/3} e_i.
$$

Summing the last inequality for $i = 1, \ldots, n$ gives us

$$
4g(G) \leq 3 \left(\frac{2}{9}\right)^{1/3} f(G)^{1/3} f(G), \quad \text{i.e.} \quad g(G)^3 \leq \frac{3}{32} f(G)^4.
$$

The constant $c = 3/32$ is the best possible. Indeed, in a complete graph $C_n$ it holds that $g(K_n)^3/f(K_n)^4 = \binom{n}{4}^3 \binom{n}{5}^{-4} \to \frac{3}{32}$ as $n \to \infty$.

Remark. Let $N_k$ be the number of complete $k$-subgraphs in a finite graph $G$. Continuing inductively, one can prove that $N_{k+1} \leq \frac{k!}{(k+1)!} N_k^k$. 

16. Note that $\triangle ANM \sim \triangle ABC$ and consequently $AM \neq AN$. Since $OM = ON$, it follows that $OR$ is a perpendicular bisector of $MN$. Thus, $R$ is the common point of the median of $MN$ and the bisector of $\angle MAN$. Then it follows from a well-known fact that $R$ lies on the circumcircle of $\triangle AMN$. Let $K$ be the intersection of $AR$ and $BC$. We then have $\angle MRA = \angle MNA = \angle ABK$ and $\angle NRA = \angle NMA = \angle ACK$, from which we conclude that $\triangle RMBK$ and $\triangle RNCK$ are cyclic. Thus $K$ is the desired intersection of the circumcircles of $\triangle BMR$ and $\triangle CNR$ and it indeed lies on $BC$.

17. Let $H$ be the reflection of $G$ about $AB$ ($GH \parallel \ell$). Let $M$ be the intersection of $AB$ and $\ell$. Since $\angle FEA = \angle FMA = 90^\circ$, it follows that $\triangle AEMF$ is cyclic and hence $\angle DFE = \angle BAE = \angle DEF$. The last equality holds because $DE$ is tangent to $\Gamma$. It follows that $DE = DF$ and hence $DF^2 = DE^2 = DC \cdot DA$ (the power of $D$ with respect to $\Gamma$). It then follows that $\angle DCF = \angle DFA = \angle HGA = \angle HCA$. Thus it follows that $H$ lies on $CF$ as desired.
18. It is important to note that since $\beta < \gamma$, $\angle ADC = 90^\circ - \gamma + \beta$ is acute. It is elementary that $\angle CAO = 90^\circ - \beta$. Let $X$ and $Y$ respectively be the intersections of $FE$ and $GH$ with $AD$. We trivially get $X \in EF \perp AD$ and $\triangle AGH \cong \triangle ACB$. Consequently, $\angle GAY = \angle OAB = 90^\circ - \gamma = 90^\circ - \angle AGY$. Hence, $GH \perp AD$ and thus $GH \parallel FE$. That $EFGH$ is a rectangle is now equivalent to $FX = GY$ and $EX = HY$. We have that $GY = AG \sin \gamma = AC \sin \gamma$ and $FX = AF \sin \gamma$ (since $\angle AFX = \gamma$). Thus,

$$FX = GY \Leftrightarrow CF = AF = AC \Leftrightarrow \angle AFC = 60^\circ \Leftrightarrow \angle ADC = 30^\circ.$$  

Since $\angle ADC = 180^\circ - \angle DCA - \angle DAC = 180^\circ - \gamma - (90^\circ - \beta)$, it immediately follows that $FX = GY \Leftrightarrow \gamma - \beta = 60^\circ$. We similarly obtain $EX = HY \Leftrightarrow \gamma - \beta = 60^\circ$, proving the statement of the problem.

19. Assume first that the points $A, B, C, D$ are concyclic. Let the lines $BP$ and $DP$ meet the circumcircle of $ABCD$ again at $E$ and $F$, respectively.

Then it follows from the given conditions that $\overline{AB} = \overline{CF}$ and $\overline{AD} = \overline{CE}$; hence $BF \parallel AC$ and $DE \parallel AC$. Therefore $BFED$ and $BFAC$ are isosceles trapezoids and thus $P = BE \cap DF$ lies on the common bisector of segments $BF, ED, AC$. Hence $AP = CP$.

Assume in turn that $AP = CP$. Let $P$ w.l.o.g. lie in the triangles $ACD$ and $BCD$. Let $BP$ and $DP$ meet $AC$ at $K$ and $L$, respectively. The points $A$ and $C$ are isogonal conjugates with respect to $\triangle BDP$, which implies that $\angle APK = \angle CPL$. Since $AP = CP$, we infer that $K$ and $L$ are symmetric with respect to the perpendicular bisector $p$ of $AC$. Let $E$ be the reflection of $D$ in $p$. Then $E$ lies on the line $BP$, and the triangles $APD$ and $CPE$ are congruent. Thus $\angle BDC = \angle ADP = \angle BEC$, which means that the points $B, C, E, D$ are concyclic. Moreover, $A, C, E, D$ are also concyclic. Hence, $ABCD$ is a cyclic quadrilateral.

20. We first establish the following lemma.

**Lemma.** Let $ABCD$ be an isosceles trapezoid with bases $AB$ and $CD$. The diagonals $AC$ and $BD$ intersect at $S$. Let $M$ be the midpoint of $BC$, and let the bisector of the angle $BSC$ intersect $BC$ at $N$. Then $\angle AMD = \angle AND$.

**Proof.** It suffices to show that the points $A, D, M, N$ are concyclic. The statement is trivial for $AD \parallel BC$. Let us now assume that $AD$ and $BC$ meet at $X$, and let $XA = XB = a$, $XC = XD = b$. Since $SN$ is the bisector of $\angle CSB$, we have

$$\frac{a - XN}{XN - b} = \frac{BN}{CN} = \frac{BS}{CS} = \frac{AB}{CD} = \frac{a}{b},$$  

and an easy computation yields $XN = \frac{2ab}{a+b}$. We also have $XM = \frac{a+b}{2}$; hence $XM \cdot XN = XA \cdot XD$. Therefore $A, D, M, N$ are concyclic, as needed.
Denote by $C_i$ the midpoint of the side $A_iA_{i+1}$, $i = 1, \ldots, n - 1$. By definition $C_1 = B_1$ and $C_{n-1} = B_{n-1}$. Since $A_1A_i \parallel A_{i+1}A_n$ for $i = 2, \ldots, n - 2$, it follows from the lemma that $\angle A_1B_iA_n = \angle A_1C_iA_n$ for all $i$.

The sum in consideration thus equals $\angle A_1C_1A_n + \angle A_1C_2A_n + \cdots + \angle A_1C_{n-1}A_n$. Moreover, the triangles $A_1C_1A_n$ and $A_{n+2-i}C_1A_{n+1-i}$ are congruent (a rotation about the center of the $n$-gon carries the first one to the second), and consequently

$$\angle A_1C_iA_n = \angle A_{n+2-i}C_1A_{n+1-i}$$

for $i = 2, \ldots, n - 1$.

Hence $\Sigma = \angle A_1C_1A_n + \angle A_1C_2A_n + \cdots + \angle A_3C_1A_2 = \angle A_1C_1A_2 = 180^\circ$.

21. Let $ABC$ be the triangle of maximum area $S$ contained in $\mathcal{P}$ (it exists because of compactness of $\mathcal{P}$). Draw parallels to $BC, CA, AB$ through $A, B, C$, respectively, and denote the triangle thus obtained by $A_1B_1C_1$ ($A \in B_1C_1$, etc.). Since each triangle with vertices in $\mathcal{P}$ has area at most $S$, the entire polygon $\mathcal{P}$ is contained in $A_1B_1C_1$.

Next, draw lines of support of $\mathcal{P}$ parallel to $BC, CA, AB$ and not intersecting the triangle $ABC$. They determine a convex hexagon $U_aV_aU_bV_bU_cV_c$ containing $\mathcal{P}$, with $V_b, U_c \in B_1C_1$, $V_c, U_a \in C_1A_1$, $V_a, U_b \in A_1B_1$. Each of the line segments $U_aV_a, U_bV_b, U_cV_c$ contains points of $\mathcal{P}$. Choose such points $A_0, B_0, C_0$ on $U_aV_a, U_bV_b, U_cV_c$, respectively. The convex hexagon $AC_0B_0A_0C_0B_0$ is contained in $\mathcal{P}$, because the latter is convex. We prove that $AC_0B_0A_0C_0B_0$ has area at least $3/4$ the area of $\mathcal{P}$.

Let $x, y, z$ denote the areas of triangles $U_aBC$, $U_bCA$, and $U_cAB$. Then $S_1 = S_{AC_0B_0A_0C_0B_0} = S + x + y + z$. On the other hand, the triangle $A_1U_aV_a$ is similar to $\triangle A_1BC$ with similitude $\tau = (S - x)/S$, and hence its area is $\tau^2S = (S - x)^2/S$. Thus the area of quadrilateral $U_aV_aCB$ is $S - (S - x)^2/S = 2z - z^2/S$. Analogous formulas hold for quadrilaterals $U_bV_bAC$ and $U_cV_cBA$. Therefore

$$S_{\mathcal{P}} \leq S_{U_aV_aU_bV_bU_cV_c} = S + S_{U_aV_aCB} + S_{U_bV_bAC} + S_{U_cV_cBA}$$

$$= S + 2(x + y + z) - \frac{x^2 + y^2 + z^2}{S}$$

$$\leq S + 2(x + y + z) - \frac{(x + y + z)^2}{3S}.$$

Now $4S_1 - 3S_{\mathcal{P}} \geq S - 2(x + y + z) + (x + y + z)^2/S = (S - x - y - z)^2/S \geq 0$; i.e., $S_1 \geq 3S_{\mathcal{P}}/4$, as claimed.

22. The proof uses the following observation:
Lemma. In a triangle $ABC$, let $K, L$ be the midpoints of the sides $AC, AB$, respectively, and let the incircle of the triangle touch $BC, CA$ at $D, E$, respectively. Then the lines $KL$ and $DE$ intersect on the bisector of the angle $ABC$.

Proof. Let the bisector $\ell_b$ of $\angle ABC$ meet $DE$ at $T$. One can assume that $AB \neq BC$, or else $T \equiv K \in KL$. Note that the incenter $I$ of $\triangle ABC$ is between $B$ and $T$, and also $T \neq E$. From the triangles $BDT$ and $DEC$ we obtain $\angle ITD = \alpha/2 = \angle IAE$, which implies that $A, I, T, E$ are concyclic. Then $\angle ATB = \angle AEI = 90^\circ$. Thus $L$ is the circumcenter of $\triangle ATB$ from which $\angle LTB = \angle LBT = \angle TBC \Rightarrow LT \parallel BC \Rightarrow T \in KL$, which is what we were supposed to prove.

Let the incircles of $\triangle ABX$ and $\triangle ACX$ touch $BX$ at $D$ and $F$, respectively, and let them touch $AX$ at $E$ and $G$, respectively. Clearly, $DE$ and $FG$ are parallel. If the line $PQ$ intersects $BX$ and $AX$ at $M$ and $N$, respectively, then $MD^2 = MP \cdot MQ = MF^2$, i.e., $MD = MF$ and analogously $NE = NG$. It follows that $PQ$ is parallel to $DE$ and $FG$ and equidistant from them.

The midpoints of $AB, AC$, and $AX$ lie on the same line $m$, parallel to $BC$. Applying the lemma to $\triangle ABX$, we conclude that $DE$ passes through the common point $U$ of $m$ and the bisector of $\angle ABX$. Analogously, $FG$ passes through the common point $V$ of $m$ and the bisector of $\angle ACX$. Therefore $PQ$ passes through the midpoint $W$ of the line segment $UV$. Since $U, V$ do not depend on $X$, neither does $W$.

23. To start with, note that point $N$ is uniquely determined by the imposed properties. Indeed, $f(X) = AX/BX$ is a monotone function on both arcs $AB$ of the circumcircle of $\triangle ABM$.

Denote by $P$ and $Q$ respectively the second points of intersection of the line $EF$ with the circumcircles of $\triangle ABE$ and $\triangle ABF$. The problem is equivalent to showing that $N \in PQ$. In fact, we shall prove that $N$ coincides with the midpoint $\overline{N}$ of segment $PQ$.

The cyclic quadrilaterals $APBE$, $AQB$, and $ABCD$ yield $\angle APQ = 180^\circ - \angle APE = 180^\circ - \angle ABE = \angle ADC$ and $\angle AQP = \angle AQF = \angle ABF = \angle ACD$. It follows that $\triangle APQ \sim \triangle ADC$, and consequently $\triangle A\overline{NP} \sim \triangle AMD$. Analogously $\triangle B\overline{NP} \sim \triangle BMC$. Therefore $A\overline{N}/AM = PQ/DC = B\overline{N}/BM$, i.e., $A\overline{N}/B\overline{N} = AM/BM$. Moreover, $\angle A\overline{NB} = \angle A\overline{NP} + \angle P\overline{NB} = \angle AMD + \angle BMC = 180^\circ - \angle AMB$, which means that point $\overline{N}$ lies on
the circumcircle of \( \triangle AMB \). By the uniqueness of \( N \), we conclude that \( \overline{N} \equiv N \), which completes the solution.

24. Setting \( m = an \) we reduce the given equation to \( m/\tau(m) = a \).
Let us show that for \( a = p^{\alpha-1} \) the above equation has no solutions in \( \mathbb{N} \) if \( p > 3 \) is a prime. Assume to the contrary that \( m \in \mathbb{N} \) is such that \( m = p^{\alpha-1} \tau(m) \). Then \( p^{\alpha-1} \mid m \), so we may set \( m = p^\alpha k \), where \( \alpha, k \in \mathbb{N} \), \( \alpha \geq p - 1 \), and \( p \nmid k \). Let \( k = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \) be the decomposition of \( k \) into primes. Then \( \tau(k) = (\alpha_1 + 1) \cdots (\alpha_r + 1) \) and \( \tau(m) = (\alpha + 1)\tau(k) \). Our equation becomes 

\[ p^{\alpha-p+1}k = (\alpha + 1)\tau(k). \tag{1} \]

We observe that \( \alpha \neq p - 1 \): otherwise the RHS would be divisible by \( p \) and the LHS would not be so. It follows that \( \alpha \geq p \), which also easily implies that \( p^{\alpha-p+1} \geq \frac{p}{p+1}(\alpha + 1) \).

Furthermore, since \( \alpha + 1 \) cannot be divisible by \( p^{\alpha-p+1} \) for any \( \alpha \geq p \), it follows that \( p \mid \tau(k) \). Thus if \( p \mid \tau(k) \), then at least one \( \alpha_i + 1 \) is divisible by \( p \) and consequently \( \alpha_i \geq p - 1 \) for some \( i \). Hence \( k \geq \frac{p}{\alpha_i+1}\tau(k) \geq \frac{2^{p-1}}{p} \tau(k) \).

But then we have 

\[ p^{\alpha-p+1}k \geq \frac{p}{p+1}(\alpha + 1) \cdot \frac{2^{p-1}}{p} \tau(k) > (\alpha + 1)\tau(k), \]

contradicting (1). Therefore (1) has no solutions in \( \mathbb{N} \).

\textit{Remark.} There are many other values of \( a \) for which the considered equation has no solutions in \( \mathbb{N} \); for example, \( a = 6p \) for a prime \( p \geq 5 \).

25. Let \( n \) be a natural number. For each \( k = 1, 2, \ldots, n \), the number \( (k, n) \) is a divisor of \( n \). Consider any divisor \( d \) of \( n \). If \( (k, n) = n/d \), then \( k = nl/d \) for some \( l \in \mathbb{N} \), and \( (k, n) = (l, d)\cdot n/d \), which implies that \( l \) is coprime to \( d \) and \( l \leq d \). It follows that \( (k, n) \) is equal to \( n/d \) for exactly \( \varphi(d) \) natural numbers \( k \leq n \).

Therefore

\[ \psi(n) = \sum_{k=1}^{n} (k, n) = \sum_{d|n} \varphi(d) \frac{n}{d} = n \sum_{d|n} \frac{\varphi(d)}{d}. \tag{1} \]

(a) Let \( n, m \) be coprime. Then each divisor \( f \) of \( mn \) can be uniquely expressed as \( f = de \), where \( d \mid n \) and \( e \mid m \). We now have by (1)

\begin{align*}
\psi(mn) &= mn \sum_{f|mn} \frac{\varphi(f)}{f} = mn \sum_{d|n, e|m} \frac{\varphi(de)}{de} \\
&= mn \sum_{d|n, e|m} \frac{\varphi(d)}{d}\frac{\varphi(e)}{e} = \left( n \sum_{d|n} \frac{\varphi(d)}{d} \right) \left( m \sum_{e|m} \frac{\varphi(e)}{e} \right) \\
&= \psi(m)\psi(n).
\end{align*}
(b) Let \( n = p^k \), where \( p \) is a prime and \( k \) a positive integer. According to (1),

\[
\psi(n) = \frac{n}{\varphi(n)} = \sum_{i=0}^{k} \frac{\varphi(p^i)}{p^i} = 1 + \frac{k(p-1)}{p}.
\]

Setting \( p = 2 \) and \( k = 2(a-1) \) we obtain \( \psi(n) = an \) for \( n = 2^{2(a-1)} \).

(c) We note that \( \psi(p^k) = p^{k+1} \) if \( p \) is a prime. Hence, if \( a \) has an odd prime factor \( p \) and \( a_1 = a/p \), then \( x = p^{2^{2a_1-2}} \) is a solution of \( \psi(x) = ax \) different from \( x = 2^{2a-2} \).

Now assume that \( a = 2^k \) for some \( k \in \mathbb{N} \). Suppose \( x = 2^a y \) is a positive integer such that \( \psi(x) = 2^k x \). Then \( 2^{a+k} y = \psi(x) = \psi(2^a) \psi(y) = (\alpha+2)2^{\alpha-1} \psi(y) \), i.e., \( 2^{k+1} y = (\alpha+2) \psi(y) \). We notice that for each odd \( y \), \( \psi(y) \) is (by definition) the sum of an odd number of odd summands and therefore odd. It follows that \( \psi(y) \mid y \). On the other hand, \( \psi(y) > y \) for \( y > 1 \), so we must have \( y = 1 \). Consequently \( a = 2^{k+1} - 2 = 2a-2 \), giving us the unique solution \( x = 2^{2a-2} \).

Thus \( \psi(x) = ax \) has a unique solution if and only if \( a \) is a power of 2.

26. For \( m = n = 1 \) we obtain that \( f(1)^2 + f(1) \) divides \( (1^2 + 1)^2 = 4 \), from which we find that \( f(1) = 1 \).

Next, we show that \( f(p-1) = p-1 \) for each prime \( p \). By the hypothesis for \( m = 1 \) and \( n = p-1 \), \( f(p-1) + 1 \) divides \( p^2 \), so \( f(p-1) \) equals either \( p-1 \) or \( p^2 - 1 \). If \( f(p-1) = p^2 - 1 \), then \( f(1) + f(p-1)^2 = p^4 - 2p^2 + 2 \) divides \( (1+(p-1)^2)^2 < p^4 - 2p^2 + 2 \), giving a contradiction. Hence \( f(p-1) = p-1 \).

Let us now consider an arbitrary \( n \in \mathbb{N} \). By the hypothesis for \( m = p-1 \), \( A = f(n) + (p-1)^2 \) divides \( (n+(p-1)^2)^2 \equiv (n-f(n))^2 \) (mod \( A \)), and hence \( A \) divides \( (n-f(n))^2 \) for any prime \( p \). Taking \( p \) large enough, we can obtain \( A \) to be greater than \( (n-f(n))^2 \), which implies that \( (n-f(n))^2 = 0 \), i.e., \( f(n) = n \) for every \( n \).

27. Set \( a = 1 \) and assume that \( b \in \mathbb{N} \) is such that \( b^2 \equiv b + 1 \) (mod \( m \)). An easy induction gives us \( x_n \equiv b^n \) (mod \( m \)) for all \( n \in \mathbb{N}_0 \). Moreover, \( b \) is obviously coprime to \( m \), and hence each \( x_n \) is coprime to \( m \).

It remains to show the existence of \( b \). The congruence \( b^2 \equiv b + 1 \) (mod \( m \)) is equivalent to \( (2b-1)^2 \equiv 5 \) (mod \( m \)). Taking \( 2b-1 \equiv 2k \), i.e., \( b \equiv 2k^2 + k - 2 \) (mod \( m \)), does the job.

Remark. A desired \( b \) exists whenever 5 is a quadratic residue modulo \( m \), in particular, when \( m \) is a prime of the form \( 10k \pm 1 \).

28. If \( n \) is divisible by 20, then every multiple of \( n \) has two last digits even and hence it is not alternate. We shall show that any other \( n \) has an alternate multiple.

(i) Let \( n \) be coprime to 10. For each \( k \) there exists a number \( A_k(n) = \frac{10 \cdots 010 \cdots 01}{10^{m_k-1}} = \frac{10^{m_k}}{10^{m_k-1}} \) (\( m \in \mathbb{N} \)) that is divisible by \( n \) (by Euler’s theorem, choose \( m = \varphi[n(10^k - 1)] \)). In particular, \( A_2(n) \) is alternate.
(ii) Let \( n = 2 \cdot 5^r \cdot n_1 \), where \( r \geq 1 \) and \( (n_1, 10) = 1 \). We shall show by induction that, for each \( k \), there exists an alternative \( k \)-digit odd number \( M_k \) that is divisible by \( 5^k \). Choosing the number \( 10A_2r(n_1)M_{2r} \) will then solve this case, since it is clearly alternate and divisible by \( n \).

We can trivially choose \( M_1 = 5 \). Let there be given an alternate \( r \)-digit multiple \( M_r \) of \( 5^r \), and let \( c \in \{0, 1, 2, 3, 4\} \) be such that \( M_r/5^r \equiv -c \cdot 2^r \pmod{5} \). Then the \((r+1)\)digit numbers \( M_r + c \cdot 10^r \) and \( M_r + (5 + c) \cdot 10^r \) are respectively equal to \( 5^r(M_r/5^r + 2^r \cdot c) \) and \( 5^r(M_r/5^r + 2^r \cdot c + 5 \cdot 2^r) \), and hence they are divisible by \( 5^{r+1} \) and exactly one of them is alternate: we set it to be \( M_{r+1} \).

(iii) Let \( n = 2^r \cdot n_1 \), where \( r \geq 1 \) and \( (n_1, 10) = 1 \). We show that there exists an alternate \( 2r \)-digit number \( N_r \) that is divisible by \( 2^{2r+1} \). Choosing the number \( A_2r(n_1)N_r \) will then solve this case.

We choose \( N_1 = 16 \), and given \( N_r \), we can prove that one of \( N_r + m \cdot 10^r \), for \( m \in \{10, 12, 14, 16\} \), is divisible by \( 2^{2r+3} \) and therefore suitable for \( N_{r+1} \). Indeed, for \( N_r = 2^{2r+1}d \) we have \( N_r + m \cdot 10^r = 2^{2r+1}(d + 5^r m/2) \) and \( d + 5^r m/2 \equiv 0 \pmod{4} \) has a solution \( m/2 \in \{5, 6, 7, 8\} \) for each \( d \) and \( r \).

Remark. The idea is essentially the same as in (SL94-24).

29. Let \( S_n = \{x \in \mathbb{N} \mid x \leq n, \; n \mid x^2 - 1\} \). It is easy to check that \( P_n \equiv 1 \pmod{n} \) for \( n = 2 \) and \( P_n \equiv -1 \pmod{n} \) for \( n \in \{3, 4\} \), so from now on we assume \( n > 4 \).

We note that if \( x \in S_n \), then also \( n-x \in S_n \) and \( (x,n) = 1 \). Thus \( S_n \) splits into pairs \( \{x, n-x\} \), where \( x \in S_n \) and \( x \leq n/2 \). In each of these pairs the product of elements gives remainder \(-1\) upon division by \( n \). Therefore \( P_n \equiv (-1)^m \), where \( S_n \) has \( 2m \) elements. It remains to find the parity of \( m \).

Suppose first that \( n > 4 \) is divisible by \( 4 \). Whenever \( x \in S_n \), the numbers \( |n/2-x|, \; n-x, \; n-|n/2-x| \) also belong to \( S_n \) (indeed, \( n \mid (n/2-x)^2 - 1 = n^2/4 - nx + x^2 - 1 \) because \( n \mid n^2/4 \), etc.). In this way the set \( S_n \) splits into four-element subsets \( \{x, n/2-x, n/2+x, n-x\} \), where \( x \in S_n \) and \( x < n/4 \) (elements of these subsets are different for \( x \neq n/4 \), and \( n/4 \) doesn’t belong to \( S_n \) for \( n > 4 \)). Therefore \( m = |S_n|/2 \) is even and \( P_n \equiv 1 \pmod{m} \).

Now let \( n \) be odd. If \( n \mid x^2 - 1 = (x-1)(x+1) \), then there exist natural numbers \( a, b \) such that \( ab = n \), \( a \mid x-1 \), \( b \mid x+1 \). Obviously \( a \) and \( b \) are coprime. Conversely, given any odd \( a, b \in \mathbb{N} \) such that \( (a, b) = 1 \) and \( ab = n \), by the Chinese remainder theorem there exists \( x \in \{1, 2, \ldots, n-1\} \) such that \( a \mid x-1 \) and \( b \mid x+1 \). This gives a bijection between all ordered pairs \( (a, b) \) with \( ab = n \) and \( (a, b) = 1 \) and the elements of \( S_n \). Now if \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) is the decomposition of \( n \) into primes, the number of pairs \( (a, b) \) is equal to \( 2^k \) (since for every \( i \), either \( p_i^{\alpha_i} \mid a \) or \( p_i^{\alpha_i} \mid b \)), and hence
$m = 2^{k-1}$. Thus $P_n \equiv -1 \pmod{n}$ if $n$ is a power of an odd prime, and $P_n \equiv 1$ otherwise.

Finally, let $n$ be even but not divisible by 4. Then $x \in S_n$ if and only if $x$ or $n-x$ belongs to $S_{n/2}$ and $x$ is odd. Since $n/2$ is odd, for each $x \in S_{n/2}$ either $x$ or $x+n/2$ belongs to $S_n$, and by the case of $n$ odd we have $S_n \equiv \pm 1 \pmod{n/2}$, depending on whether or not $n/2$ is a power of a prime. Since $S_n$ is odd, it follows that $P_n \equiv -1 \pmod{n}$ if $n/2$ is a power of a prime, and $P_n \equiv 1$ otherwise.

Second solution. Obviously $S_n$ is closed under multiplication modulo $n$. This implies that $S_n$ with multiplication modulo $n$ is a subgroup of $\mathbb{Z}_n$, and therefore there exist elements $a_1 = -1, a_2, \ldots, a_k \in S_n$ that generate $S_n$. In other words, since the $a_i$ are of order two, $S_n$ consists of products $\prod_{i \in A} a_i$, where $A$ runs over all subsets of $\{1, 2, \ldots, k\}$. Thus $S_n$ has $2^k$ elements, and the product of these elements equals $P_n \equiv (a_1 a_2 \cdots a_k)^{2^{k-1}} \pmod{n}$. Since $a_i^2 \equiv 1 \pmod{n}$, it follows that $P_n \equiv 1$ if $k \geq 2$, i.e., if $|S_n| > 2$. Otherwise $P_n \equiv -1 \pmod{n}$.

We note that $|S_n| > 2$ is equivalent to the existence of $a \in S_n$ with $1 < a < n-1$. It is easy to find that such an $a$ exists if and only if neither of $n, n/2$ is a power of an odd prime.

30. We shall denote by $k$ the given circle with diameter $p^n$.

Let $A, B$ be lattice points (i.e., points with integer coordinates). We shall denote by $\mu(AB)$ the exponent of the highest power of $p$ that divides the integer $AB^2$. We observe that if $S$ is the area of a triangle $ABC$ where $A, B, C$ are lattice points, then $2S$ is an integer. According to Heron’s formula and the formula for the circumradius, a triangle $ABC$ whose circumcenter has diameter $p^n$ satisfies

$$2AB^2BC^2 + 2BC^2CA^2 + 2CA^2AB^2 - AB^4 - BC^4 - CA^4 = 16S^2 \quad (1)$$

and

$$AB^2 \cdot BC^2 \cdot CA^2 = (2S)^2p^{2n}. \quad (2)$$

Lemma 1. Let $A, B, C$ be lattice points on $k$. If none of $AB^2, BC^2, CA^2$ is divisible by $p^{n+1}$, then $\mu(AB), \mu(BC), \mu(CA)$ are $0, n, n$ in some order.

Proof. Let $k = \min\{\mu(AB), \mu(BC), \mu(CA)\}$. By (1), $(2S)^2$ is divisible by $p^{2k}$. Together with (2), this gives us $\mu(AB) + \mu(BC) + \mu(CA) = 2k + 2n$. On the other hand, if none of $AB, BC, CA$ is divisible by $p^{n+1}$, then $\mu(AB) + \mu(BC) + \mu(CA) \leq k + 2n$. Therefore $k = 0$ and the remaining two of $\mu(AB), \mu(BC), \mu(CA)$ are equal to $n$.

Lemma 2. Among every four lattice points on $k$, there exist two, say $M, N$, such that $\mu(MN) \geq n+1$.

Proof. Assume that this doesn’t hold for some points $A, B, C, D$ on $k$.

By Lemma 1, $\mu$ for some of the segments $AB, AC, \ldots, CD$ is 0, say $\mu(AC) = 0$. It easily follows by Lemma 1 that then $\mu(BD) = 0$ and $\mu(AB) = \mu(BC) = \mu(CD) = \mu(DA) = n$. Let $a, b, c, d, e, f \in \mathbb{N}$ be
such that $AB^2 = p^n a$, $BC^2 = p^n b$, $CD^2 = p^n c$, $DA^2 = p^n d$, $AC^2 = e$, $BD^2 = f$. By Ptolemy’s theorem we have $\sqrt{ef} = p^n \left(\sqrt{ac} + \sqrt{bd}\right)$.

Taking squares, we get that $\frac{ef}{p^{2n}} = (\sqrt{ac} + \sqrt{bd})^2 = ac + bd + 2\sqrt{abcd}$ is rational and hence an integer. It follows that $ef$ is divisible by $p^{2n}$, a contradiction.

Now we consider eight lattice points $A_1, A_2, \ldots, A_8$ on $k$. We color each segment $A_iA_j$ red if $\mu(A_iA_j) > n$ and black otherwise, and thus obtain a graph $G$. The degree of a point $X$ will be the number of red segments with an endpoint in $X$. We distinguish three cases:

(i) There is a point, say $A_8$, whose degree is at most 1. We may suppose w.l.o.g. that $A_8A_7$ is red and $A_8A_1, \ldots, A_8A_6$ black. By a well-known fact, the segments joining vertices $A_1, A_2, \ldots, A_6$ determine either a red triangle, in which case there is nothing to prove, or a black triangle, say $A_1A_2A_3$. But in the latter case the four points $A_1, A_2, A_3, A_8$ do not determine any red segment, a contradiction to Lemma 2.

(ii) All points have degree 2. Then the set of red segments partitions into cycles. If one of these cycles has length 3, then the proof is complete. If all the cycles have length at least 4, then we have two possibilities: two 4-cycles, say $A_1A_2A_3A_4$ and $A_5A_6A_7A_8$, or one 8-cycle, $A_1A_2 \ldots A_8$. In both cases, the four points $A_1, A_3, A_5, A_7$ do not determine any red segment, a contradiction.

(iii) There is a point of degree at least 3, say $A_1$. Suppose that $A_1A_2$, $A_1A_3$, and $A_1A_4$ are red. We claim that $A_2, A_3, A_4$ determine at least one red segment, which will complete the solution. If not, by Lemma 1, $\mu(A_2A_3), \mu(A_3A_4), \mu(A_4A_2)$ are $n, n, 0$ in some order. Assuming w.l.o.g. that $\mu(A_2A_3) = 0$, denote by $S$ the area of triangle $A_1A_2A_3$. Now by formula (1), $2S$ is not divisible by $p$. On the other hand, since $\mu(A_1A_2) \geq n + 1$ and $\mu(A_1A_3) \geq n + 1$, it follows from (2) that $2S$ is divisible by $p$, a contradiction.
A

Notation and Abbreviations

A.1 Notation

We assume familiarity with standard elementary notation of set theory, algebra, logic, geometry (including vectors), analysis, number theory (including divisibility and congruences), and combinatorics. We use this notation liberally.

We assume familiarity with the basic elements of the game of chess (the movement of pieces and the coloring of the board).

The following is notation that deserves additional clarification.

- $B(A,B,C)$, $A - B - C$: indicates the relation of betweenness, i.e., that $B$ is between $A$ and $C$ (this automatically means that $A, B, C$ are different collinear points).
- $A = l_1 \cap l_2$: indicates that $A$ is the intersection point of the lines $l_1$ and $l_2$.
- $AB$: line through $A$ and $B$, segment $AB$, length of segment $AB$ (depending on context).
- $[AB]$: ray starting in $A$ and containing $B$.
- $(AB)$: ray starting in $A$ and containing $B$, but without the point $A$.
- $(AB)$: open interval $AB$, set of points between $A$ and $B$.
- $[AB]$: closed interval $AB$, segment $AB$, $(AB) \cup \{A,B\}$.
- $(AB)$: semiopen interval $AB$, closed at $B$ and open at $A$, $(AB) \cup \{B\}$.
  The same bracket notation is applied to real numbers, e.g., $[a,b) = \{x \mid a \leq x < b\}$.
- $ABC$: plane determined by points $A, B, C$, triangle $ABC$ ($\triangle ABC$) (depending on context).
- $[AB, C]$: half-plane consisting of line $AB$ and all points in the plane on the same side of $AB$ as $C$. 
\( (AB, C) \): \([AB, C]\) without the line \( AB \).

\( \langle \overrightarrow{a}, \overrightarrow{b} \rangle, \overrightarrow{a} \cdot \overrightarrow{b} \): scalar product of \( \overrightarrow{a} \) and \( \overrightarrow{b} \).

\( a, b, c, \alpha, \beta, \gamma \): the respective sides and angles of triangle \( ABC \) (unless otherwise indicated).

\( k(O, r) \): circle \( k \) with center \( O \) and radius \( r \).

\( d(A, p) \): distance from point \( A \) to line \( p \).

\( S_{A_1A_2\ldots A_n} \): area of \( n \)-gon \( A_1A_2\ldots A_n \) (special case for \( n = 3 \), \( S_{ABC} \): area of \( \triangle ABC \)).

\( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \): the sets of natural, integer, rational, real, complex numbers (respectively).

\( \mathbb{Z}_n \): the ring of residues modulo \( n \), \( n \in \mathbb{N} \).

\( \mathbb{Z}_p \): the field of residues modulo \( p \), \( p \) being prime.

\( \mathbb{Z}[x], \mathbb{R}[x] \): the rings of polynomials in \( x \) with integer and real coefficients respectively.

\( R^* \): the set of nonzero elements of a ring \( R \).

\( R[\alpha], R(\alpha) \), where \( \alpha \) is a root of a quadratic polynomial in \( R[x] \): \( \{a + b\alpha \mid a, b \in R\} \).

\( X_0 \): \( X \cup \{0\} \) for \( X \) such that \( 0 \notin X \).

\( X^+, X^-, aX + b, aX + bY \): \( \{x \mid x \in X, x > 0\} \), \( \{x \mid x \in X, x < 0\} \), \( \{ax + b \mid x \in X\} \), \( \{ax + by \mid x \in X, y \in Y\} \) (respectively) for \( X, Y \subseteq \mathbb{R}, a, b \in \mathbb{R} \).

\( [x], \lfloor x \rfloor \): the greatest integer smaller than or equal to \( x \).

\( \lceil x \rceil \): the smallest integer greater than or equal to \( x \).

The following is notation simultaneously used in different concepts (depending on context).

\( |AB|, |x|, |S| \): the distance between two points \( AB \), the absolute value of the number \( x \), the number of elements of the set \( S \) (respectively).

\( (x, y), (m, n), (a, b) \): (ordered) pair \( x \) and \( y \), the greatest common divisor of integers \( m \) and \( n \), the open interval between real numbers \( a \) and \( b \) (respectively).

A.2 Abbreviations

We tried to avoid using nonstandard notations and abbreviations as much as possible. However, one nonstandard abbreviation stood out as particularly convenient:
○ w.l.o.g.: without loss of generality.

Other abbreviations include:

○ RHS: right-hand side (of a given equation).
○ LHS: left-hand side (of a given equation).
○ QM, AM, GM, HM: the quadratic mean, the arithmetic mean, the geometric mean, the harmonic mean (respectively).
○ gcd, lcm: greatest common divisor, least common multiple (respectively).
○ i.e.: in other words.
○ e.g.: for example.
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References

44. N.D. Kazarinoff, *Geometric Inequalities*, Mathematical Association of America (MAA), 1975.
References


